Section 1.2 Finding Limits Graphically and Numerically

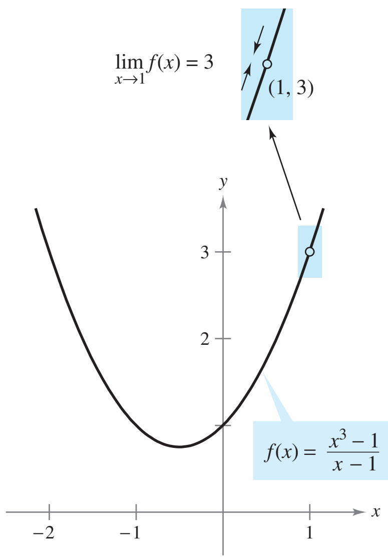
- **Estimate a limit using a numerical or graphical approach.**
- **Learn different ways that a limit can fail to exist.**
- **Study and use a formal definition of limit.**

An Introduction to Limits

Suppose you are asked to sketch the graph of the function f given by

$$f(x) = \frac{x^3 - 1}{x - 1}, \qquad x \neq 1.$$

For all values other than $x = 1$, you can use standard curve-sketching techniques. However, at $x = 1$, it is not clear what to expect. To get an idea of the behavior of the graph of f near $x = 1$, you can use two sets of x-values—one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.



The limit of $f(x)$ as x approaches 1 is 3.
Figure 1.5

x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.

The graph of f is a parabola that has a gap at the point $(1, 3)$, as shown in Figure 1.5. Although x cannot equal 1, you can move arbitrarily close to 1, and as a result $f(x)$ moves arbitrarily close to 3. Using limit notation, you can write

$$\lim_{x\to 1} f(x) = 3.$$ This is read as "the limit of $f(x)$ as x approaches 1 is 3."

This discussion leads to an informal description of a limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the **limit** of $f(x)$, as x approaches c, is L. This limit is written as

$$\lim_{x\to c} f(x) = L.$$

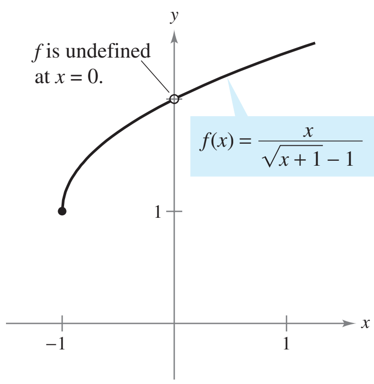
EXPLORATION

The discussion above gives an example of how you can estimate a limit *numerically* by constructing a table and *graphically* by drawing a graph. Estimate the following limit numerically by completing the table.

$$\lim_{x\to 2} \frac{x^2 - 3x + 2}{x - 2}$$

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	?	?	?	?	?	?	?	?	?

Then use a graphing utility to estimate the limit graphically.



The limit of $f(x)$ as x approaches 0 is 2.
Figure 1.6

EXAMPLE 1 Estimating a Limit Numerically

Evaluate the function $f(x) = x/\left(\sqrt{x+1} - 1\right)$ at several points near $x = 0$ and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

Solution The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. → ← x approaches 0 from the right.

x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. → ← $f(x)$ approaches 2.

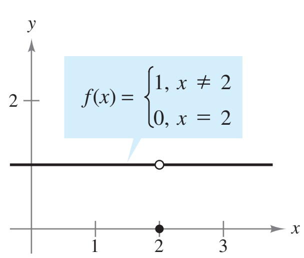
From the results shown in the table, you can estimate the limit to be 2. This limit is reinforced by the graph of f (see Figure 1.6).

In Example 1, note that the function is undefined at $x = 0$ and yet $f(x)$ appears to be approaching a limit as x approaches 0. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

EXAMPLE 2 Finding a Limit

Find the limit of $f(x)$ as x approaches 2 where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2. \end{cases}$$



The limit of $f(x)$ as x approaches 2 is 1.
Figure 1.7

Solution Because $f(x) = 1$ for all x other than $x = 2$, you can conclude that the limit is 1, as shown in Figure 1.7. So, you can write

$$\lim_{x \to 2} f(x) = 1.$$

The fact that $f(2) = 0$ has no bearing on the existence or value of the limit as x approaches 2. For instance, if the function were defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

the limit would be the same.

So far in this section, you have been estimating limits numerically and graphically. Each of these approaches produces an estimate of the limit. In Section 1.3, you will study analytic techniques for evaluating limits. Throughout the course, try to develop a habit of using this three-pronged approach to problem solving.

1. Numerical approach — Construct a table of values.
2. Graphical approach — Draw a graph by hand or using technology.
3. Analytic approach — Use algebra or calculus.

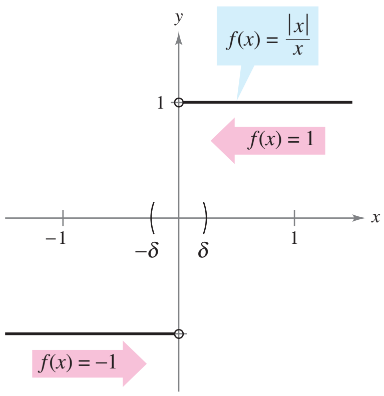
Limits That Fail to Exist

In the next three examples you will examine some limits that fail to exist.

EXAMPLE 3 Behavior That Differs from the Right and Left

Show that the limit does not exist.

$$\lim_{x\to 0} \frac{|x|}{x}$$



$\lim_{x\to 0} f(x)$ does not exist.

Figure 1.8

Solution Consider the graph of the function $f(x) = |x|/x$. From Figure 1.8, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \qquad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \qquad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$. Specifically, if δ (the lowercase Greek letter *delta*) is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $|x|/x$ as shown.

$(-\delta, 0)$ — Negative x-values yield $|x|/x = -1$.

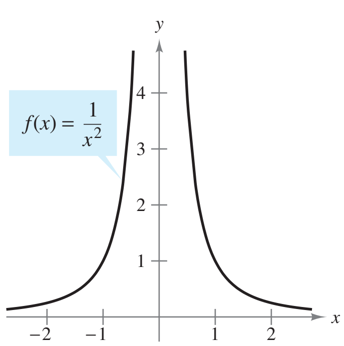
$(0, \delta)$ — Positive x-values yield $|x|/x = 1$.

This implies that the limit does not exist.

EXAMPLE 4 Unbounded Behavior

Discuss the existence of the limit

$$\lim_{x\to 0} \frac{1}{x^2}.$$



$\lim_{x\to 0} f(x)$ does not exist.

Figure 1.9

Solution Let $f(x) = 1/x^2$. In Figure 1.9, you can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

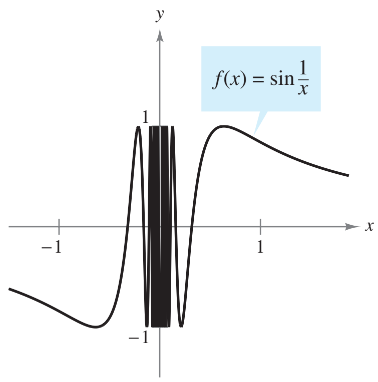
Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a real number L as x approaches 0, you can conclude that the limit does not exist.

EXAMPLE 5 Oscillating Behavior

Discuss the existence of the limit $\lim_{x\to 0} \sin \frac{1}{x}$.



$\lim_{x\to 0} f(x)$ does not exist.
Figure 1.10

Solution Let $f(x) = \sin(1/x)$. In Figure 1.10, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

x	$2/\pi$	$2/3\pi$	$2/5\pi$	$2/7\pi$	$2/9\pi$	$2/11\pi$	$x \to 0$
sin (1/x)	1	-1	1	-1	1	-1	Limit does not exist.

Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

There are many other interesting functions that have unusual limit behavior. An often cited one is the *Dirichlet function*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational.} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

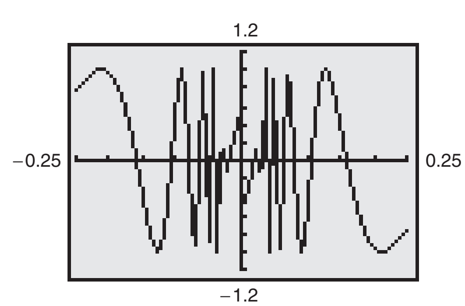
Because this function has *no limit* at any real number c, it is *not continuous* at any real number c. You will study continuity more closely in Section 1.4.

The Granger Collection

Peter Gustav Dirichlet (1805–1859)

In the early development of calculus, the definition of a function was much more restricted than it is today, and "functions" such as the Dirichlet function would not have been considered. The modern definition of function was given by the German mathematician Peter Gustav Dirichlet.

TECHNOLOGY PITFALL When you use a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you can't always trust the pictures that graphing utilities draw. If you use a graphing utility to graph the function in Example 5 over an interval containing 0, you will most likely obtain an incorrect graph such as that shown in Figure 1.11. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.



Incorrect graph of $f(x) = \sin(1/x)$.
Figure 1.11

indicates that in the HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

A Formal Definition of Limit

Let's take another look at the informal description of a limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, written as

$$\lim_{x \to c} f(x) = L.$$

At first glance, this description looks fairly technical. Even so, it is informal because exact meanings have not yet been given to the two phrases

"$f(x)$ becomes arbitrarily close to L"

and

"x approaches c."

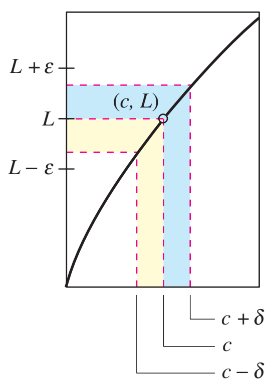
The first person to assign mathematically rigorous meanings to these two phrases was Augustin-Louis Cauchy. His ε-δ **definition of limit** is the standard used today.

In Figure 1.12, let ε (the lowercase Greek letter *epsilon*) represent a (small) positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

Similarly, the phrase "x approaches c" means that there exists a positive number δ such that x lies in either the interval $(c - \delta, c)$ or the interval $(c, c + \delta)$. This fact can be concisely expressed by the double inequality

$$0 < |x - c| < \delta.$$



The ε-δ definition of the limit of $f(x)$ as x approaches c
Figure 1.12

The first inequality

$$0 < |x - c| \qquad \text{The distance between } x \text{ and } c \text{ is more than 0.}$$

expresses the fact that $x \neq c$. The second inequality

$$|x - c| < \delta \qquad x \text{ is within } \delta \text{ units of } c.$$

says that x is within a distance δ of c.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

NOTE Throughout this text, the expression

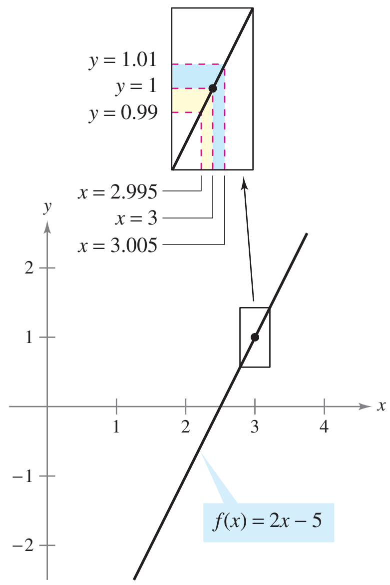
$$\lim_{x \to c} f(x) = L$$

implies two statements—the limit exists *and* the limit is L.

FOR FURTHER INFORMATION For more on the introduction of rigor to calculus, see "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

Some functions do not have limits as $x \to c$, but those that do cannot have two different limits as $x \to c$. That is, *if the limit of a function exists, it is unique* (see Exercise 69).

The next three examples should help you develop a better understanding of the ε-δ definition of limit.



The limit of $f(x)$ as x approaches 3 is 1.
Figure 1.13

EXAMPLE 6 Finding a δ for a Given ε

Given the limit

$$\lim_{x \to 3} (2x - 5) = 1$$

find δ such that $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$.

Solution In this problem, you are working with a given value of ε—namely, $\varepsilon = 0.01$. To find an appropriate δ, notice that

$$|(2x - 5) - 1| = |2x - 6| = 2|x - 3|.$$

Because the inequality $|(2x - 5) - 1| < 0.01$ is equivalent to $2|x - 3| < 0.01$, you can choose $\delta = \frac{1}{2}(0.01) = 0.005$. This choice works because

$$0 < |x - 3| < 0.005$$

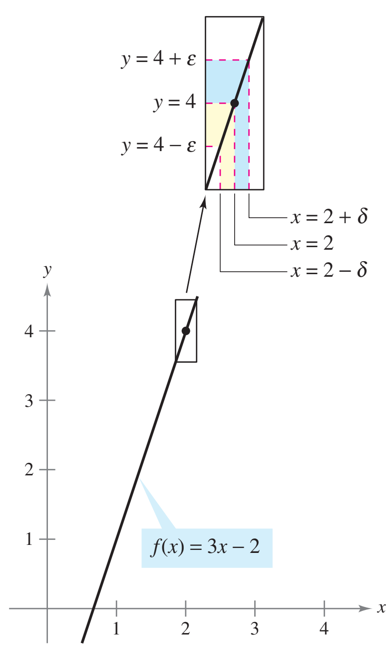
implies that

$$|(2x - 5) - 1| = 2|x - 3| < 2(0.005) = 0.01$$

as shown in Figure 1.13.

NOTE In Example 6, note that 0.005 is the *largest* value of δ that will guarantee $|(2x - 5) - 1| < 0.01$ whenever $0 < |x - 3| < \delta$. Any *smaller* positive value of δ would also work.

In Example 6, you found a δ-value for a *given* ε. This does not prove the existence of the limit. To do that, you must prove that you can find a δ for any ε, as shown in the next example.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.14

EXAMPLE 7 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x \to 2} (3x - 2) = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(3x - 2) - 4| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Because your choice of δ depends on ε, you need to establish a connection between the absolute values $|(3x - 2) - 4|$ and $|x - 2|$.

$$|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$$

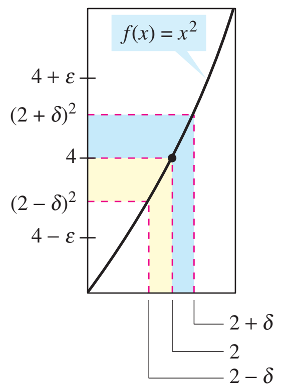
So, for a given $\varepsilon > 0$ you can choose $\delta = \varepsilon/3$. This choice works because

$$0 < |x - 2| < \delta = \frac{\varepsilon}{3}$$

implies that

$$|(3x - 2) - 4| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

as shown in Figure 1.14.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

EXAMPLE 8 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x\to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \text{ whenever } 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval $(1, 3)$, you know that $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15.

Throughout this chapter you will use the ε-δ definition of limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of limit.

Exercises for Section 1.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\lim_{x\to 2} \dfrac{x - 2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

2. $\lim_{x\to 2} \dfrac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\lim_{x\to 0} \dfrac{\sqrt{x + 3} - \sqrt{3}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

4. $\lim_{x\to -3} \dfrac{\sqrt{1 - x} - 2}{x + 3}$

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$						

5. $\lim_{x\to 3} \dfrac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\lim_{x\to 4} \dfrac{[x/(x + 1)] - (4/5)}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

7. $\lim_{x\to 0} \dfrac{\sin x}{x}$

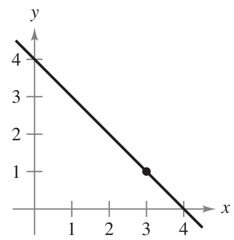
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

8. $\lim_{x\to 0} \dfrac{\cos x - 1}{x}$

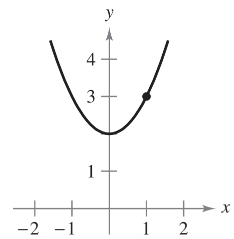
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

In Exercises 9–18, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

9. $\lim_{x\to 3}(4 - x)$

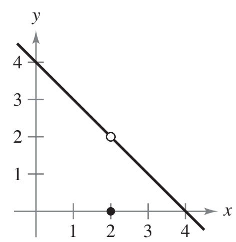


10. $\lim_{x\to 1}(x^2 + 2)$



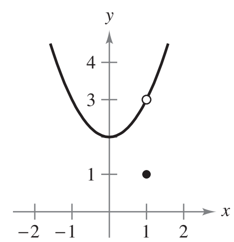
11. $\lim_{x\to 2} f(x)$

$$f(x) = \begin{cases} 4 - x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

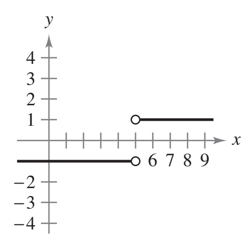


12. $\lim_{x\to 1} f(x)$

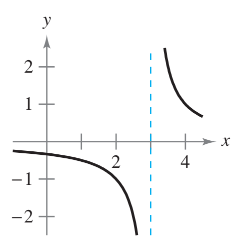
$$f(x) = \begin{cases} x^2 + 2, & x \neq 1 \\ 1, & x = 1 \end{cases}$$



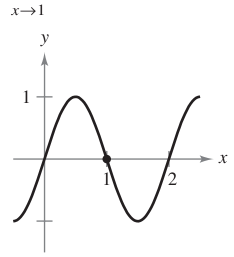
13. $\lim_{x\to 5} \dfrac{|x - 5|}{x - 5}$



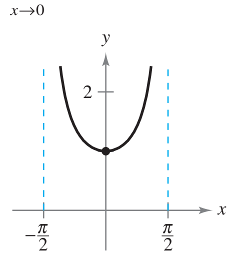
14. $\lim_{x\to 3} \dfrac{1}{x - 3}$



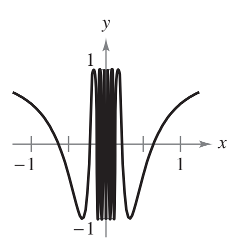
15. $\lim_{x\to 1} \sin \pi x$



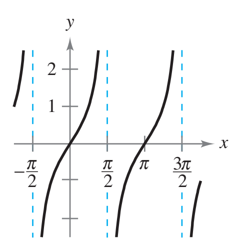
16. $\lim_{x\to 0} \sec x$



17. $\lim_{x\to 0} \cos \dfrac{1}{x}$



18. $\lim_{x\to \pi/2} \tan x$



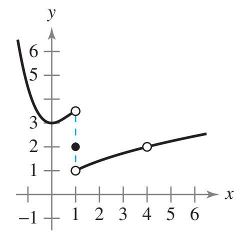
In Exercises 19 and 20, use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.

19. (a) $f(1)$

(b) $\lim_{x\to 1} f(x)$

(c) $f(4)$

(d) $\lim_{x\to 4} f(x)$



20. (a) $f(-2)$

(b) $\lim_{x\to -2} f(x)$

(c) $f(0)$

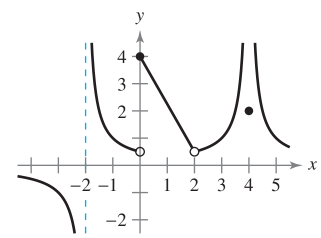
(d) $\lim_{x\to 0} f(x)$

(e) $f(2)$

(f) $\lim_{x\to 2} f(x)$

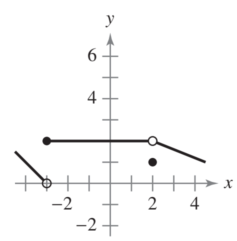
(g) $f(4)$

(h) $\lim_{x\to 4} f(x)$

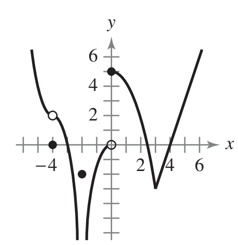


In Exercises 21 and 22, use the graph of f to identify the values of c for which $\lim_{x\to c} f(x)$ exists.

21.



22.



In Exercises 23 and 24, sketch the graph of f. Then identify the values of c for which $\lim_{x\to c} f(x)$ exists.

23. $f(x) = \begin{cases} x^2, & x \leq 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \geq 4 \end{cases}$

24. $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \leq x \leq \pi \\ \cos x, & x > \pi \end{cases}$

In Exercises 25 and 26, sketch a graph of a function f that satisfies the given values. (There are many correct answers.)

25. $f(0)$ is undefined.

$\lim_{x \to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x \to 2} f(x) = 3$

26. $f(-2) = 0$

$f(2) = 0$

$\lim_{x \to -2} f(x) = 0$

$\lim_{x \to 2} f(x)$ does not exist.

27. ***Modeling Data*** The cost of a telephone call between two cities is \$0.75 for the first minute and \$0.50 for each additional minute or fraction thereof. A formula for the cost is given by

$$C(t) = 0.75 - 0.50[\![-(t - 1)]\!]$$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \le x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \le 5$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find

$$\lim_{t \to 3.5} C(t).$$

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

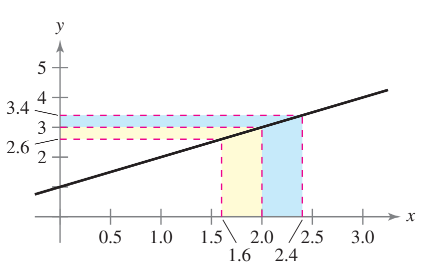
t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

28. Repeat Exercise 27 for

$$C(t) = 0.35 - 0.12[\![-(t - 1)]\!].$$

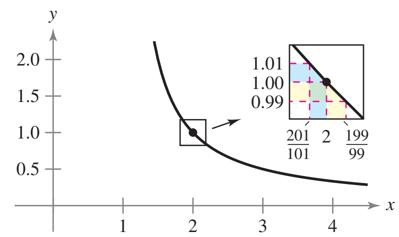
29. The graph of $f(x) = x + 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.4$.



30. The graph of

$$f(x) = \frac{1}{x - 1}$$

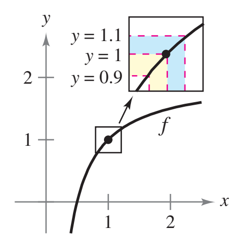
is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 1| < 0.01$.



31. The graph of

$$f(x) = 2 - \frac{1}{x}$$

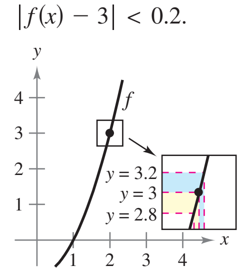
is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$ then $|f(x) - 1| < 0.1$.



32. The graph of

$$f(x) = x^2 - 1$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.2$.



In Exercises 33–36, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

33. $\lim_{x \to 2} (3x + 2)$

34. $\lim_{x \to 4} \left(4 - \frac{x}{2}\right)$

35. $\lim_{x \to 2} (x^2 - 3)$

36. $\lim_{x \to 5} (x^2 + 4)$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

In Exercises 37–48, find the limit L. Then use the ε-δ definition to prove that the limit is L.

37. $\lim_{x \to 2} (x + 3)$

38. $\lim_{x \to -3} (2x + 5)$

39. $\lim_{x \to -4} \left(\frac{1}{2}x - 1\right)$

40. $\lim_{x \to 1} \left(\frac{2}{3}x + 9\right)$

41. $\lim_{x \to 6} 3$

42. $\lim_{x \to 2} (-1)$

43. $\lim_{x \to 0} \sqrt[3]{x}$

44. $\lim_{x \to 4} \sqrt{x}$

45. $\lim_{x \to -2} |x - 2|$

46. $\lim_{x \to 3} |x - 3|$

47. $\lim_{x \to 1} (x^2 + 1)$

48. $\lim_{x \to -3} (x^2 + 3x)$

Writing **In Exercises 49–52, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

49. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

$\lim_{x \to 4} f(x)$

50. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x \to 3} f(x)$

51. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x \to 9} f(x)$

52. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x \to 3} f(x)$

Writing About Concepts

53. Write a brief description of the meaning of the notation $\lim_{x \to 8} f(x) = 25$.

54. If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.

55. If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.

Writing About Concepts *(continued)*

56. Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.

57. ***Jewelry*** A jeweler resizes a ring so that its inner circumference is 6 centimeters.

(a) What is the radius of the ring?

(b) If the ring's inner circumference can vary between 5.5 centimeters and 6.5 centimeters, how can the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

58. ***Sports*** A sporting goods manufacturer designs a golf ball having a volume of 2.48 cubic inches.

(a) What is the radius of the golf ball?

(b) If the ball's volume can vary between 2.45 cubic inches and 2.51 cubic inches, how can the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

59. Consider the function $f(x) = (1 + x)^{1/x}$. Estimate the limit

$$\lim_{x \to 0} (1 + x)^{1/x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

60. Consider the function

$$f(x) = \frac{|x + 1| - |x - 1|}{x}.$$

Estimate

$$\lim_{x \to 0} \frac{|x + 1| - |x - 1|}{x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

61. ***Graphical Analysis*** The statement

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(2 - \delta, 2 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

62. ***Graphical Analysis*** The statement

$$\lim_{x \to 3} \frac{x^2 - 3x}{x - 3}$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 3| < \delta$, then

$$\left| \frac{x^2 - 3x}{x - 3} - 3 \right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left| \frac{x^2 - 3x}{x - 3} - 3 \right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(3 - \delta, 3 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

True or False? **In Exercises 63–66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

64. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.

65. If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.

66. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

67. Consider the function $f(x) = \sqrt{x}$.

(a) Is $\lim_{x \to 0.25} \sqrt{x} = 0.5$ a true statement? Explain.

(b) Is $\lim_{x \to 0} \sqrt{x} = 0$ a true statement? Explain.

68. ***Writing*** The definition of limit on page 52 requires that f is a function defined on an open interval containing c, except possibly at c. Why is this requirement necessary?

69. Prove that if the limit of $f(x)$ as $x \to c$ exists, then the limit must be unique. [*Hint:* Let

$$\lim_{x \to c} f(x) = L_1 \quad \text{and} \quad \lim_{x \to c} f(x) = L_2$$

and prove that $L_1 = L_2$.]

70. Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of limit to prove that $\lim_{x \to c} f(x) = mc + b$.

71. Prove that $\lim_{x \to c} f(x) = L$ is equivalent to $\lim_{x \to c} [f(x) - L] = 0$.

72. (a) Given that

$$\lim_{x \to 0} (3x + 1)(3x - 1)x^2 + 0.01 = 0.01$$

prove that there exists an open interval (a, b) containing 0 such that $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ for all $x \neq 0$ in (a, b).

(b) Given that $\lim_{x \to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such that $g(x) > 0$ for all $x \neq c$ in (a, b).

73. ***Programming*** Use the programming capabilities of a graphing utility to write a program for approximating $\lim_{x \to c} f(x)$.

Assume the program will be applied only to functions whose limits exist as x approaches c. Let $y_1 = f(x)$ and generate two lists whose entries form the ordered pairs

$$(c \pm [0.1]^n, f(c \pm [0.1]^n))$$

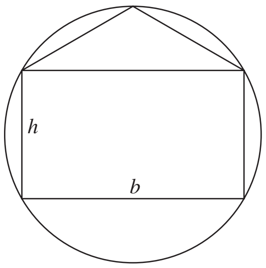
for $n = 0, 1, 2, 3,$ and 4.

74. ***Programming*** Use the program you created in Exercise 73 to approximate the limit

$$\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4}.$$

Putnam Exam Challenge

75. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?



76. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

These problems were composed by the Committee on the Putnam Prize Competition.