

Section 2.3

Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

In Section 2.2 you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

NOTE A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products involving three or more factors.

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Proof Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned} \frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x) + f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[f(x + \Delta x) \frac{g(x + \Delta x) - g(x)}{\Delta x} + g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[f(x + \Delta x) \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} \left[g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} f(x + \Delta x) \cdot \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} g(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

THE PRODUCT RULE

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx dy$ (as being negligible) and obtained the differential form $x dy + y dx$. This derivation resulted in the traditional form of the Product Rule.

(Source: *The History of Mathematics* by David M. Burton)

Note that $\lim_{\Delta x \rightarrow 0} f(x + \Delta x) = f(x)$ because f is given to be differentiable and therefore is continuous.

The Product Rule can be extended to cover products involving more than two factors. For example, if f , g , and h are differentiable functions of x , then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

For instance, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned} \frac{dy}{dx} &= 2x \sin x \cos x + x^2 \cos x \cos x + x^2 \sin x (-\sin x) \\ &= 2x \sin x \cos x + x^2(\cos^2 x - \sin^2 x). \end{aligned}$$

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$ with the derivative in Example 1.

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$\begin{aligned}
 h'(x) &= \overbrace{(3x - 2x^2)}^{\text{First}} \overbrace{\frac{d}{dx}[5 + 4x]}^{\text{Derivative of second}} + \overbrace{(5 + 4x)}^{\text{Second}} \overbrace{\frac{d}{dx}[3x - 2x^2]}^{\text{Derivative of first}} && \text{Apply Product Rule.} \\
 &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\
 &= (12x - 8x^2) + (15 - 8x - 16x^2) \\
 &= -24x^2 + 4x + 15
 \end{aligned}$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$\begin{aligned}
 D_x[(3x - 2x^2)(5 + 4x)] &= D_x[-8x^3 + 2x^2 + 15x] \\
 &= -24x^2 + 4x + 15.
 \end{aligned}$$

In the next example, you must use the Product Rule.

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

Solution

$$\begin{aligned}
 \frac{d}{dx}[3x^2 \sin x] &= 3x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[3x^2] && \text{Apply Product Rule.} \\
 &= 3x^2 \cos x + (\sin x)(6x) \\
 &= 3x^2 \cos x + 6x \sin x \\
 &= 3x(x \cos x + 2 \sin x)
 \end{aligned}$$

EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\begin{aligned}
 \frac{dy}{dx} &= \overbrace{(2x) \left(\frac{d}{dx}[\cos x] \right)}^{\text{Product Rule}} + \overbrace{(\cos x) \left(\frac{d}{dx}[2x] \right)}^{\text{Constant Multiple Rule}} - 2 \frac{d}{dx}[\sin x] \\
 &= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x) \\
 &= -2x \sin x
 \end{aligned}$$

NOTE In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

The Quotient Rule

THEOREM 2.8 The Quotient Rule

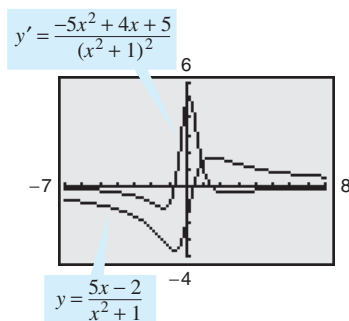
The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Proof As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] &= \lim_{\Delta x \rightarrow 0} \frac{\frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} && \text{Definition of derivative} \\ &= \lim_{\Delta x \rightarrow 0} \frac{g(x)f(x + \Delta x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{g(x)f(x + \Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)} \\ &= \frac{\lim_{\Delta x \rightarrow 0} \frac{g(x)[f(x + \Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \rightarrow 0} \frac{f(x)[g(x + \Delta x) - g(x)]}{\Delta x}}{\lim_{\Delta x \rightarrow 0} [g(x)g(x + \Delta x)]} \\ &= \frac{g(x) \left[\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] - f(x) \left[\lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \right]}{\lim_{\Delta x \rightarrow 0} [g(x)g(x + \Delta x)]} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \end{aligned}$$

TECHNOLOGY A graphing utility can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative
Figure 2.22

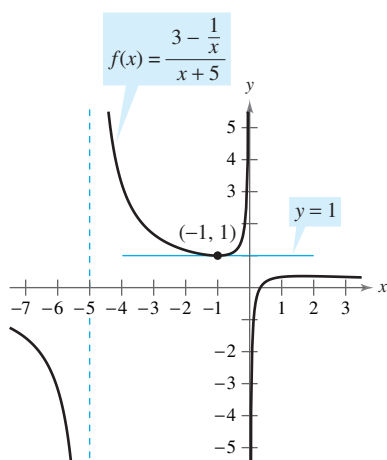
Note that $\lim_{\Delta x \rightarrow 0} g(x + \Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous.

EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \frac{5x - 2}{x^2 + 1}$.

Solution

$$\begin{aligned} \frac{d}{dx} \left[\frac{5x - 2}{x^2 + 1} \right] &= \frac{(x^2 + 1) \frac{d}{dx} [5x - 2] - (5x - 2) \frac{d}{dx} [x^2 + 1]}{(x^2 + 1)^2} && \text{Apply Quotient Rule.} \\ &= \frac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2} \\ &= \frac{(5x^2 + 5) - (10x^2 - 4x)}{(x^2 + 1)^2} \\ &= \frac{-5x^2 + 4x + 5}{(x^2 + 1)^2} \end{aligned}$$



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.

Figure 2.23

Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \frac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Solution Begin by rewriting the function.

$$\begin{aligned}
 f(x) &= \frac{3 - (1/x)}{x + 5} && \text{Write original function.} \\
 &= \frac{x\left(3 - \frac{1}{x}\right)}{x(x + 5)} && \text{Multiply numerator and denominator by } x. \\
 &= \frac{3x - 1}{x^2 + 5x} && \text{Rewrite.} \\
 f'(x) &= \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} && \text{Quotient Rule} \\
 &= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2} \\
 &= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} && \text{Simplify.}
 \end{aligned}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \quad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 2.23.

Not every quotient needs to be differentiated by the Quotient Rule. For example, each quotient in the next example can be considered as the product of a constant times a function of x . In such cases it is more convenient to use the Constant Multiple Rule.

EXAMPLE 6 Using the Constant Multiple Rule

Original Function	Rewrite	Differentiate	Simplify
a. $y = \frac{x^2 + 3x}{6}$	$y = \frac{1}{6}(x^2 + 3x)$	$y' = \frac{1}{6}(2x + 3)$	$y' = \frac{2x + 3}{6}$
b. $y = \frac{5x^4}{8}$	$y = \frac{5}{8}x^4$	$y' = \frac{5}{8}(4x^3)$	$y' = \frac{5}{2}x^3$
c. $y = \frac{-3(3x - 2x^2)}{7x}$	$y = -\frac{3}{7}(3 - 2x)$	$y' = -\frac{3}{7}(-2)$	$y' = \frac{6}{7}$
d. $y = \frac{9}{5x^2}$	$y = \frac{9}{5}(x^{-2})$	$y' = \frac{9}{5}(-2x^{-3})$	$y' = -\frac{18}{5x^3}$

NOTE To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

In Section 2.2, the Power Rule was proved only for the case where the exponent n is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

EXAMPLE 7 Proof of the Power Rule (Negative Integer Exponents)

If n is a negative integer, there exists a positive integer k such that $n = -k$. So, by the Quotient Rule, you can write

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^k}\right] \\ &= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} && \text{Quotient Rule and Power Rule} \\ &= \frac{0 - kx^{k-1}}{x^{2k}} \\ &= -kx^{-k-1} \\ &= nx^{n-1}. && n = -k\end{aligned}$$

So, the Power Rule

$$D_x[x^n] = nx^{n-1} \quad \text{Power Rule}$$

is valid for any integer. In Exercise 75 in Section 2.5, you are asked to prove the case for which n is any rational number.

Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

THEOREM 2.9 Derivatives of Trigonometric Functions

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \sec^2 x & \frac{d}{dx}[\cot x] &= -\csc^2 x \\ \frac{d}{dx}[\sec x] &= \sec x \tan x & \frac{d}{dx}[\csc x] &= -\csc x \cot x\end{aligned}$$

Proof Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} && \text{Apply Quotient Rule.} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 89).



EXAMPLE 8 Differentiating Trigonometric Functions

NOTE Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

<u>Function</u>	<u>Derivative</u>
a. $y = x - \tan x$	$\frac{dy}{dx} = 1 - \sec^2 x$
b. $y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$ $= (\sec x)(1 + x \tan x)$

EXAMPLE 9 Different Forms of a Derivative

Differentiate both forms of $y = \frac{1 - \cos x}{\sin x} = \csc x - \cot x$.

Solution

$$\begin{aligned} \text{First form: } y &= \frac{1 - \cos x}{\sin x} \\ y' &= \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x} \\ &= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x} \\ &= \frac{1 - \cos x}{\sin^2 x} \end{aligned}$$

$$\begin{aligned} \text{Second form: } y &= \csc x - \cot x \\ y' &= -\csc x \cot x + \csc^2 x \end{aligned}$$

To show that the two derivatives are equal, you can write

$$\begin{aligned} \frac{1 - \cos x}{\sin^2 x} &= \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) \\ &= \csc^2 x - \csc x \cot x. \end{aligned}$$

The summary below shows that much of the work in obtaining a simplified form of a derivative occurs *after* differentiating. Note that two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

	<i>f'(x) After Differentiating</i>	<i>f'(x) After Simplifying</i>
Example 1	$(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$	$-24x^2 + 4x + 15$
Example 3	$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$	$-2x \sin x$
Example 4	$\frac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$	$\frac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$
Example 5	$\frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$	$\frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$
Example 9	$\frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$	$\frac{1 - \cos x}{\sin^2 x}$

Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

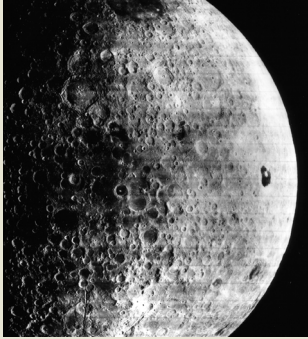
$$\begin{aligned} s(t) & \text{ Position function} \\ v(t) = s'(t) & \text{ Velocity function} \\ a(t) = v'(t) = s''(t) & \text{ Acceleration function} \end{aligned}$$

NOTE: The second derivative of f is the derivative of the first derivative of f .

The function given by $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative**. You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as follows.

$$\begin{aligned} \text{First derivative:} & \quad y', & f'(x), & \frac{dy}{dx}, & \frac{d}{dx}[f(x)], & D_x[y] \\ \text{Second derivative:} & \quad y'', & f''(x), & \frac{d^2y}{dx^2}, & \frac{d^2}{dx^2}[f(x)], & D_x^2[y] \\ \text{Third derivative:} & \quad y''', & f'''(x), & \frac{d^3y}{dx^3}, & \frac{d^3}{dx^3}[f(x)], & D_x^3[y] \\ \text{Fourth derivative:} & \quad y^{(4)}, & f^{(4)}(x), & \frac{d^4y}{dx^4}, & \frac{d^4}{dx^4}[f(x)], & D_x^4[y] \\ & \quad \vdots & & & & \\ \text{nth derivative:} & \quad y^{(n)}, & f^{(n)}(x), & \frac{d^ny}{dx^n}, & \frac{d^n}{dx^n}[f(x)], & D_x^n[y] \end{aligned}$$



NASA

THE MOON

The moon's mass is 7.349×10^{22} kilograms, and Earth's mass is 5.976×10^{24} kilograms. The moon's radius is 1737 kilometers, and Earth's radius is 6378 kilometers. Because the gravitational force on the surface of a planet is directly proportional to its mass and inversely proportional to the square of its radius, the ratio of the gravitational force on Earth to the gravitational force on the moon is

$$\frac{(5.976 \times 10^{24})/6378^2}{(7.349 \times 10^{22})/1737^2} \approx 6.03.$$

EXAMPLE 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds. What is the ratio of Earth's gravitational force to the moon's?

Solution To find the acceleration, differentiate the position function twice.

$$\begin{aligned} s(t) &= -0.81t^2 + 2 & \text{Position function} \\ s'(t) &= -1.62t & \text{Velocity function} \\ s''(t) &= -1.62 & \text{Acceleration function} \end{aligned}$$

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\begin{aligned} \frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} &= \frac{-9.8}{-1.62} \\ &\approx 6.05. \end{aligned}$$

Exercises for Section 2.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, use the Product Rule to differentiate the function.

1. $g(x) = (x^2 + 1)(x^2 - 2x)$
2. $f(x) = (6x + 5)(x^3 - 2)$
3. $h(t) = \sqrt[3]{t}(t^2 + 4)$
4. $g(s) = \sqrt{s}(4 - s^2)$
5. $f(x) = x^3 \cos x$
6. $g(x) = \sqrt{x} \sin x$

In Exercises 7–12, use the Quotient Rule to differentiate the function.

7. $f(x) = \frac{x}{x^2 + 1}$
8. $g(t) = \frac{t^2 + 2}{2t - 7}$
9. $h(x) = \frac{\sqrt[3]{x}}{x^3 + 1}$
10. $h(s) = \frac{s}{\sqrt{s} - 1}$
11. $g(x) = \frac{\sin x}{x^2}$
12. $f(t) = \frac{\cos t}{t^3}$

In Exercises 13–18, find $f'(x)$ and $f'(c)$.

Function	Value of c
13. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$	$c = 0$
14. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$	$c = 1$
15. $f(x) = \frac{x^2 - 4}{x - 3}$	$c = 1$
16. $f(x) = \frac{x + 1}{x - 1}$	$c = 2$
17. $f(x) = x \cos x$	$c = \frac{\pi}{4}$
18. $f(x) = \frac{\sin x}{x}$	$c = \frac{\pi}{6}$

In Exercises 19–24, complete the table without using the Quotient Rule.

Function	Rewrite	Differentiate	Simplify
19. $y = \frac{x^2 + 2x}{3}$			
20. $y = \frac{5x^2 - 3}{4}$			
21. $y = \frac{7}{3x^3}$			
22. $y = \frac{4}{5x^2}$			
23. $y = \frac{4x^{3/2}}{x}$			
24. $y = \frac{3x^2 - 5}{7}$			

In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \frac{3 - 2x - x^2}{x^2 - 1}$
26. $f(x) = \frac{x^3 + 3x + 2}{x^2 - 1}$

27. $f(x) = x\left(1 - \frac{4}{x+3}\right)$
28. $f(x) = x^4\left(1 - \frac{2}{x+1}\right)$
29. $f(x) = \frac{2x+5}{\sqrt{x}}$
30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$
31. $h(s) = (s^3 - 2)^2$
32. $h(x) = (x^2 - 1)^2$
33. $f(x) = \frac{2 - \frac{1}{x}}{x - 3}$
34. $g(x) = x^2\left(\frac{2}{x} - \frac{1}{x+1}\right)$
35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$
36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$
37. $f(x) = \frac{x^2 + c^2}{x^2 - c^2}$, c is a constant
38. $f(x) = \frac{c^2 - x^2}{c^2 + x^2}$, c is a constant

In Exercises 39–54, find the derivative of the trigonometric function.


39. $f(t) = t^2 \sin t$
40. $f(\theta) = (\theta + 1) \cos \theta$
41. $f(t) = \frac{\cos t}{t}$
42. $f(x) = \frac{\sin x}{x}$
43. $f(x) = -x + \tan x$
44. $y = x + \cot x$
45. $g(t) = \sqrt[4]{t} + 8 \sec t$
46. $h(s) = \frac{1}{s} - 10 \csc s$
47. $y = \frac{3(1 - \sin x)}{2 \cos x}$
48. $y = \frac{\sec x}{x}$
49. $y = -\csc x - \sin x$
50. $y = x \sin x + \cos x$
51. $f(x) = x^2 \tan x$
52. $f(x) = \sin x \cos x$
53. $y = 2x \sin x + x^2 \cos x$
54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

 In Exercises 55–58, use a computer algebra system to differentiate the function.

55. $g(x) = \left(\frac{x+1}{x+2}\right)(2x-5)$
56. $f(x) = \left(\frac{x^2-x-3}{x^2+1}\right)(x^2+x+1)$
57. $g(\theta) = \frac{\theta}{1 - \sin \theta}$
58. $f(\theta) = \frac{\sin \theta}{1 - \cos \theta}$

In Exercises 59–62, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

Function	Point
59. $y = \frac{1 + \csc x}{1 - \csc x}$	$\left(\frac{\pi}{6}, -3\right)$
60. $f(x) = \tan x \cot x$	$(1, 1)$
61. $h(t) = \frac{\sec t}{t}$	$\left(\pi, -\frac{1}{\pi}\right)$
62. $f(x) = \sin x(\sin x + \cos x)$	$\left(\frac{\pi}{4}, 1\right)$

 In Exercises 63–68, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

63. $f(x) = (x^3 - 3x + 1)(x + 2)$, $(1, -3)$

64. $f(x) = (x - 1)(x^2 - 2)$, $(0, 2)$

65. $f(x) = \frac{x}{x - 1}$, $(2, 2)$

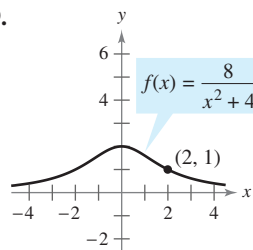
66. $f(x) = \frac{(x - 1)}{(x + 1)}$, $(2, \frac{1}{3})$

67. $f(x) = \tan x$, $(\frac{\pi}{4}, 1)$

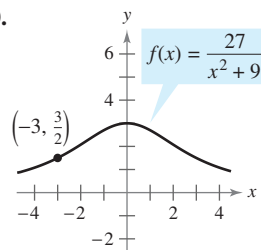
68. $f(x) = \sec x$, $(\frac{\pi}{3}, 2)$

Famous Curves In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentes*.)

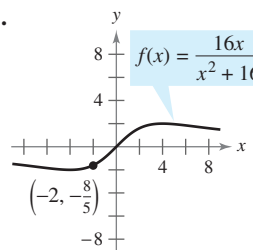
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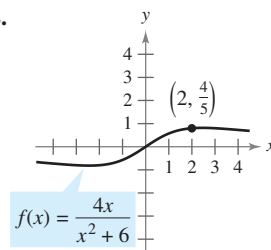
70.



71.



72.



In Exercises 73–76, determine the point(s) at which the graph of the function has a horizontal tangent line.

73. $f(x) = \frac{x^2}{x - 1}$

74. $f(x) = \frac{x^2}{x^2 + 1}$

75. $f(x) = \frac{4x - 2}{x^2}$

76. $f(x) = \frac{x - 4}{x^2 - 7}$

77. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = \frac{x + 1}{x - 1}$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

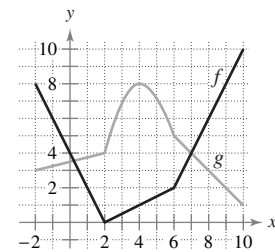
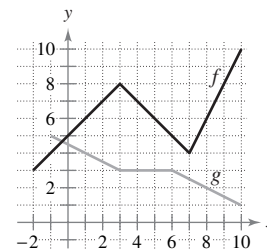
78. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = \frac{x}{x - 1}$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

In Exercises 79 and 80, verify that $f'(x) = g'(x)$, and explain the relationship between f and g .

79. $f(x) = \frac{3x}{x + 2}$, $g(x) = \frac{5x + 4}{x + 2}$

80. $f(x) = \frac{\sin x - 3x}{x}$, $g(x) = \frac{\sin x + 2x}{x}$

In Exercises 81 and 82, use the graphs of f and g . Let $p(x) = f(x)g(x)$ and $q(x) = \frac{f(x)}{g(x)}$.

81. (a) Find $p'(1)$.82. (a) Find $p'(4)$.(b) Find $q'(4)$.(b) Find $q'(7)$.

83. Area The length of a rectangle is given by $2t + 1$ and its height is \sqrt{t} , where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

84. Volume The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

85. Inventory Replenishment The ordering and transportation cost C for the components used in manufacturing a product is

$$C = 100 \left(\frac{200}{x^2} + \frac{x}{x + 30} \right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

86. Boyle's Law This law states that if the temperature of a gas remains constant, its pressure is inversely proportional to its volume. Use the derivative to show that the rate of change of the pressure is inversely proportional to the square of the volume.

87. Population Growth A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500 \left(1 + \frac{4t}{50 + t^2} \right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

128 CHAPTER 2 Differentiation

88. **Gravitational Force** Newton's Law of Universal Gravitation states that the force F between two masses, m_1 and m_2 , is

$$F = \frac{Gm_1m_2}{d^2}$$


where G is a constant and d is the distance between the masses. Find an equation that gives an instantaneous rate of change of F with respect to d . (Assume m_1 and m_2 represent moving points.)

89. Prove the following differentiation rules.

$$(a) \frac{d}{dx}[\sec x] = \sec x \tan x \quad (b) \frac{d}{dx}[\csc x] = -\csc x \cot x$$

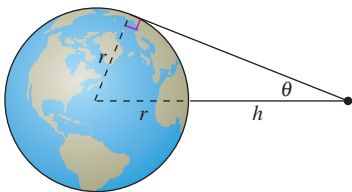
$$(c) \frac{d}{dx}[\cot x] = -\csc^2 x$$

90. **Rate of Change** Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

-  91. **Modeling Data** The table shows the numbers n (in thousands) of motor homes sold in the United States and the retail values v (in billions of dollars) of these motor homes for the years 1996 through 2001. The year is represented by t , with $t = 6$ corresponding to 1996. (Source: Recreation Vehicle Industry Association)

Year, t	6	7	8	9	10	11
n	247.5	254.5	292.7	321.2	300.1	256.8
v	6.3	6.9	8.4	10.4	9.5	8.6

- (a) Use a graphing utility to find cubic models for the number of motor homes sold $n(t)$ and the total retail value $v(t)$ of the motor homes.
 (b) Graph each model found in part (a).
 (c) Find $A = v(t)/n(t)$, then graph A . What does this function represent?
 (d) Interpret $A'(t)$ in the context of these data.
92. **Satellites** When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from Earth's surface and let r represent Earth's radius.



- (a) Show that $h = r(\csc \theta - 1)$.
 (b) Find the rate at which h is changing with respect to θ when $\theta = 30^\circ$. (Assume $r = 3960$ miles.)

In Exercises 93–98, find the second derivative of the function.

93. $f(x) = 4x^{3/2}$

94. $f(x) = x + 32x^{-2}$

95. $f(x) = \frac{x}{x-1}$

96. $f(x) = \frac{x^2 + 2x - 1}{x}$

97. $f(x) = 3 \sin x$

98. $f(x) = \sec x$

In Exercises 99–102, find the given higher-order derivative.

99. $f'(x) = x^2$, $f''(x)$

100. $f''(x) = 2 - \frac{2}{x}$, $f'''(x)$

101. $f'''(x) = 2\sqrt{x}$, $f^{(4)}(x)$

102. $f^{(4)}(x) = 2x + 1$, $f^{(6)}(x)$

Writing About Concepts

103. Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$.
 104. Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x .

In Exercises 105–108, use the given information to find $f'(2)$.

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

105. $f(x) = 2g(x) + h(x)$

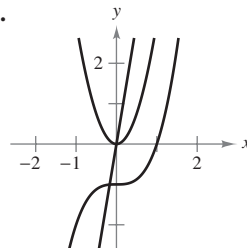
106. $f(x) = 4 - h(x)$

107. $f(x) = \frac{g(x)}{h(x)}$

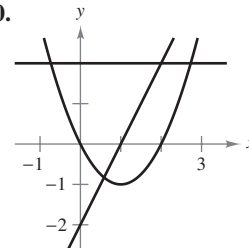
108. $f(x) = g(x)h(x)$

In Exercises 109 and 110, the graphs of f , f' , and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

109.

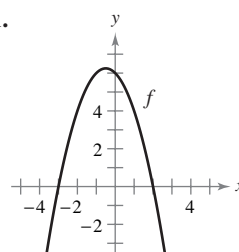


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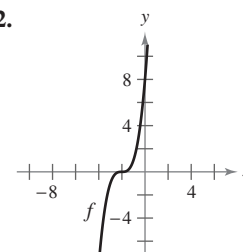


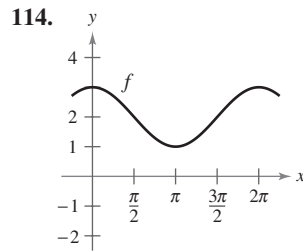
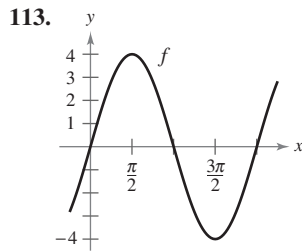
In Exercises 111–114, the graph of f is shown. Sketch the graphs of f' and f'' . To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

111.



112.





115. **Acceleration** The velocity of an object in meters per second is $v(t) = 36 - t^2$, $0 \leq t \leq 6$. Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

116. **Acceleration** An automobile's velocity starting from rest is

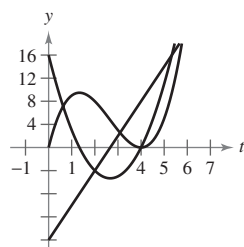
$$v(t) = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleration at (a) 5 seconds, (b) 10 seconds, and (c) 20 seconds.

117. **Stopping Distance** A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is $s(t) = -8.25t^2 + 66t$, where s is measured in feet and t is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

t	0	1	2	3	4
$s(t)$					
$v(t)$					
$a(t)$					

118. **Particle Motion** The figure shows the graphs of the position, velocity, and acceleration functions of a particle.



- (a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.
- (b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.

Finding a Pattern In Exercises 119 and 120, develop a general rule for $f^{(n)}(x)$ given $f(x)$.

119. $f(x) = x^n$

120. $f(x) = \frac{1}{x}$

121. **Finding a Pattern** Consider the function $f(x) = g(x)h(x)$.

(a) Use the Product Rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results in part (a) to write a general rule for $f^{(n)}(x)$.

122. **Finding a Pattern** Develop a general rule for $[xf(x)]^{(n)}$ where f is a differentiable function of x .

In Exercises 123 and 124, find the derivatives of the function f for $n = 1, 2, 3$, and 4. Use the results to write a general rule for $f'(x)$ in terms of n .

123. $f(x) = x^n \sin x$

124. $f(x) = \frac{\cos x}{x^n}$

Differential Equations In Exercises 125–128, verify that the function satisfies the differential equation.

<u>Function</u>	<u>Differential Equation</u>
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125. $y = \frac{1}{x}, x > 0$

$x^3 y'' + 2x^2 y' = 0$

126. $y = 2x^3 - 6x + 10$

$-y''' - xy'' - 2y' = -24x^2$

127. $y = 2 \sin x + 3$

$y'' + y = 3$

128. $y = 3 \cos x + \sin x$

$y'' + y = 0$

True or False? In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

129. If $y = f(x)g(x)$, then $dy/dx = f'(x)g'(x)$.

130. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $d^5y/dx^5 = 0$.

131. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

132. If $f(x)$ is an n th-degree polynomial, then $f^{(n+1)}(x) = 0$.

133. The second derivative represents the rate of change of the first derivative.

134. If the velocity of an object is constant, then its acceleration is zero.

135. Find a second-degree polynomial $f(x) = ax^2 + bx + c$ such that its graph has a tangent line with slope 10 at the point $(2, 7)$ and an x -intercept at $(1, 0)$.

136. Consider the third-degree polynomial

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0.$$

Determine conditions for a , b , c , and d if the graph of f has (a) no horizontal tangents, (b) exactly one horizontal tangent, and (c) exactly two horizontal tangents. Give an example for each case.

137. Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist?

138. **Think About It** Let f and g be functions whose first and second derivatives exist on an interval I . Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$

(b) $fg'' + f''g = (fg)''$