Section 4.5 Integration by Substitution

- **Use pattern recognition to find an indefinite integral.**
- **Use a change of variables to find an indefinite integral.**
- **Use the General Power Rule for Integration to find an indefinite integral.**
- **Use a change of variables to evaluate a definite integral.**
- **Evaluate a definite integral involving an even or odd function.**

Pattern Recognition

In this section you will study techniques for integrating composite functions. The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***u*-substitution.** With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$
$$= F(u) + C.$$

These results are summarized in the following theorem.

NOTE The statement of Theorem 4.12 doesn't tell how to distinguish between $f(g(x))$ and $g'(x)$ in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

THEOREM 4.12 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

STUDY TIP There are several techniques for applying substitution, each differing slightly from the others. However, you should remember that the goal is the same with every technique—*you are trying to find an antiderivative of the integrand.*

EXPLORATION

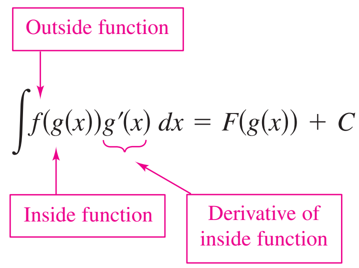
Recognizing Patterns The integrand in each of the following integrals fits the pattern $f(g(x))g'(x)$. Identify the pattern and use the result to evaluate the integral.

a. $\int 2x(x^2 + 1)^4\,dx$ **b.** $\int 3x^2\sqrt{x^3 + 1}\,dx$ **c.** $\int \sec^2 x(\tan x + 3)\,dx$

The next three integrals are similar to the first three. Show how you can multiply and divide by a constant to evaluate these integrals.

d. $\int x(x^2 + 1)^4\,dx$ **e.** $\int x^2\sqrt{x^3 + 1}\,dx$ **f.** $\int 2\sec^2 x(\tan x + 3)\,dx$

Examples 1 and 2 show how to apply Theorem 4.12 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.



EXAMPLE 1 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2 + 1)^2(2x)\,dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for Integration and Theorem 4.12, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is the integrand of the original integral.

TECHNOLOGY Try using a computer algebra system, such as *Maple*, *Derive*, *Mathematica*, *Mathcad*, or the *TI-89*, to solve the integrals given in Examples 1 and 2. Do you obtain the same antiderivatives that are listed in the examples?

EXAMPLE 2 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int 5 \cos 5x\,dx$.

Solution Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos 5x.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Cosine Rule for Integration and Theorem 4.12, you can write

$$\int \overbrace{(\cos 5x)}^{f(g(x))}\overbrace{(5)}^{g'(x)}\,dx = \sin 5x + C.$$

You can check this by differentiating $\sin 5x + C$ to obtain the original integrand.

The integrands in Examples 1 and 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern. You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

EXAMPLE 3 Multiplying and Dividing by a Constant

Find $\int x(x^2 + 1)^2\,dx$.

Solution This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2. Recognizing that $2x$ is the derivative of $x^2 + 1$, you can let $g(x) = x^2 + 1$ and supply the $2x$ as follows.

$$\int x(x^2 + 1)^2\,dx = \int (x^2 + 1)^2 \left(\frac{1}{2}\right)(2x)\,dx \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx \qquad \text{Constant Multiple Rule}$$

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C \qquad \text{Integrate.}$$

$$= \frac{1}{6}(x^2 + 1)^3 + C \qquad \text{Simplify.}$$

In practice, most people would not write as many steps as are shown in Example 3. For instance, you could evaluate the integral by simply writing

$$\int x(x^2 + 1)^2\,dx = \frac{1}{2}\int (x^2 + 1)^2\,2x\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C$$

$$= \frac{1}{6}(x^2 + 1)^3 + C.$$

NOTE Be sure you see that the *Constant* Multiple Rule applies only to *constants*. You cannot multiply and divide by a variable and then move the variable outside the integral sign. For instance,

$$\int (x^2 + 1)^2\,dx \neq \frac{1}{2x}\int (x^2 + 1)^2(2x)\,dx.$$

After all, if it were legitimate to move variable quantities outside the integral sign, you could move the entire integrand out and simplify the whole process. But the result would be incorrect.

Change of Variables

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variable technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 4.12 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

EXAMPLE 4 Change of Variables

Find $\int \sqrt{2x - 1}\,dx$.

Solution First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2\,dx$. Now, using $\sqrt{2x - 1} = \sqrt{u}$ and $dx = du/2$, substitute to obtain

$$\begin{aligned}
\int \sqrt{2x-1}\,dx &= \int \sqrt{u}\left(\frac{du}{2}\right) && \text{Integral in terms of } u\\
&= \frac{1}{2}\int u^{1/2}\,du && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C && \text{Antiderivative in terms of } u\\
&= \frac{1}{3}u^{3/2} + C && \text{Simplify.}\\
&= \frac{1}{3}(2x-1)^{3/2} + C. && \text{Antiderivative in terms of } x
\end{aligned}$$

STUDY TIP Because integration is usually more difficult than differentiation, you should always check your answer to an integration problem by differentiating. For instance, in Example 4 you should differentiate $\frac{1}{3}(2x - 1)^{3/2} + C$ to verify that you obtain the original integrand.

EXAMPLE 5 Change of Variables

Find $\int x\sqrt{2x - 1}\,dx$.

Solution As in the previous example, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as shown.

$$u = 2x - 1 \quad \Rightarrow \quad x = (u + 1)/2 \qquad \text{Solve for } x \text{ in terms of } u.$$

Now, using substitution, you obtain

$$\begin{aligned}
\int x\sqrt{2x-1}\,dx &= \int \left(\frac{u+1}{2}\right)u^{1/2}\left(\frac{du}{2}\right)\\
&= \frac{1}{4}\int (u^{3/2} + u^{1/2})\,du\\
&= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C\\
&= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.
\end{aligned}$$

To complete the change of variables in Example 5, you solved for x in terms of u. Sometimes this is very difficult. Fortunately it is not always necessary, as shown in the next example.

EXAMPLE 6 Change of Variables

Find $\int \sin^2 3x \cos 3x \, dx$.

Solution Because $\sin^2 3x = (\sin 3x)^2$, you can let $u = \sin 3x$. Then

$$du = (\cos 3x)(3)\, dx.$$

Now, because $\cos 3x\, dx$ is part of the original integral, you can write

$$\frac{du}{3} = \cos 3x\, dx.$$

Substituting u and $du/3$ in the original integral yields

$$\begin{aligned}\int \sin^2 3x \cos 3x\, dx &= \int u^2 \frac{du}{3}\\ &= \frac{1}{3}\int u^2\, du\\ &= \frac{1}{3}\left(\frac{u^3}{3}\right) + C\\ &= \frac{1}{9}\sin^3 3x + C.\end{aligned}$$

You can check this by differentiating.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{9}\sin^3 3x\right] &= \left(\frac{1}{9}\right)(3)(\sin 3x)^2(\cos 3x)(3)\\ &= \sin^2 3x \cos 3x\end{aligned}$$

Because differentiation produces the original integrand, you know that you have obtained the correct antiderivative.

STUDY TIP When making a change of variables, be sure that your answer is written using the same variables as in the original integrand. For instance, in Example 6, you should not leave your answer as

$$\tfrac{1}{9}u^3 + C$$

but rather, replace u by $\sin 3x$.

The steps used for integration by substitution are summarized in the following guidelines.

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\, dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration.** A proof of this rule follows directly from the (simple) Power Rule for Integration, together with Theorem 4.12.

THEOREM 4.13 The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1.$$

EXAMPLE 7 Substitution and the General Power Rule

a. $$\int 3(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)\,dx}^{du} = \overbrace{\frac{(3x-1)^5}{5}}^{u^5/5} + C$$

b. $$\int (2x+1)(x^2+x)\,dx = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,dx}^{du} = \overbrace{\frac{(x^2+x)^2}{2}}^{u^2/2} + C$$

c. $$\int 3x^2\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{3/2}}^{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2} + C$$

d. $$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,dx}^{du} = \overbrace{\frac{(1-2x^2)^{-1}}{-1}}^{u^{-1}/(-1)} + C = -\frac{1}{1-2x^2} + C$$

e. $$\int \cos^2 x \sin x\,dx = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,dx}^{du} = -\overbrace{\frac{(\cos x)^3}{3}}^{u^3/3} + C$$

EXPLORATION

Suppose you were asked to find one of the following integrals. Which one would you choose? Explain your reasoning.

a. $\int \sqrt{x^3+1}\,dx$ or

$\int x^2\sqrt{x^3+1}\,dx$

b. $\int \tan(3x)\sec^2(3x)\,dx$ or

$\int \tan(3x)\,dx$

Some integrals whose integrands involve quantities raised to powers cannot be found by the General Power Rule. Consider the two integrals

$$\int x(x^2+1)^2\,dx \quad \text{and} \quad \int (x^2+1)^2\,dx.$$

The substitution $u = x^2 + 1$ works in the first integral but not in the second. In the second, the substitution fails because the integrand lacks the factor x needed for du. Fortunately, *for this particular integral*, you can expand the integrand as $(x^2+1)^2 = x^4 + 2x^2 + 1$ and use the (simple) Power Rule to integrate each term.

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.12 combined with the Fundamental Theorem of Calculus.

THEOREM 4.14 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE 8 Change of Variables

Evaluate $\displaystyle\int_0^1 x(x^2 + 1)^3\,dx$.

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \implies du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\int_0^1 x(x^2 + 1)^3\,dx = \frac{1}{2}\int_0^1 (x^2 + 1)^3(2x)\,dx$$

Integration limits for x

$$= \frac{1}{2}\int_1^2 u^3\,du$$

Integration limits for u

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right)$$

$$= \frac{15}{8}.$$

Try rewriting the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown.

$$\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left[\frac{(x^2 + 1)^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$$

Notice that you obtain the same result.

EXAMPLE 9 Change of Variables

Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx.$

Solution To evaluate this integral, let $u = \sqrt{2x-1}$. Then, you obtain

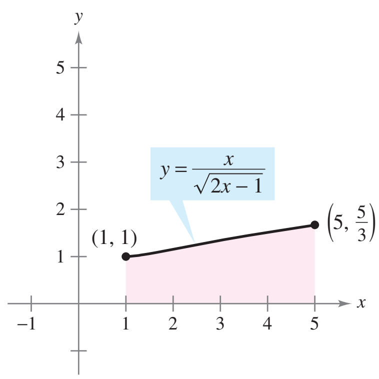
$$u^2 = 2x - 1$$
$$u^2 + 1 = 2x$$
$$\frac{u^2+1}{2} = x$$
$$u\,du = dx. \qquad \text{Differentiate each side.}$$

Before substituting, determine the new upper and lower limits of integration.

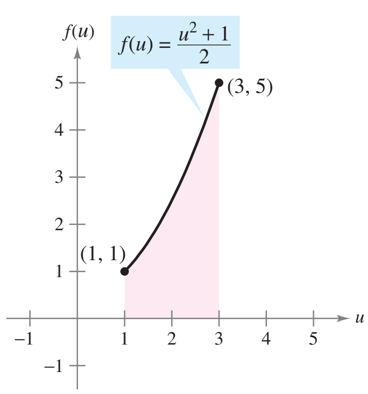
Lower Limit	*Upper Limit*
When $x = 1$, $u = \sqrt{2-1} = 1$.	When $x = 5$, $u = \sqrt{10-1} = 3$.

Now, substitute to obtain

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right) u\,du$$
$$= \frac{1}{2}\int_1^3 (u^2+1)\,du$$
$$= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3$$
$$= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right)$$
$$= \frac{16}{3}.$$



The region before substitution has an area of $\frac{16}{3}$.
Figure 4.37



The region after substitution has an area of $\frac{16}{3}$.
Figure 4.38

Geometrically, you can interpret the equation

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{u^2+1}{2}\,du$$

to mean that the two *different* regions shown in Figures 4.37 and 4.38 have the *same* area.

When evaluating definite integrals by substitution, it is possible for the upper limit of integration of the u-variable form to be smaller than the lower limit. If this happens, don't rearrange the limits. Simply evaluate as usual. For example, after substituting $u = \sqrt{1-x}$ in the integral

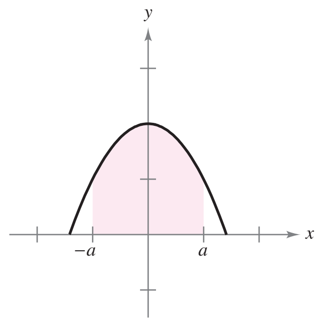
$$\int_0^1 x^2(1-x)^{1/2}\,dx$$

you obtain $u = \sqrt{1-1} = 0$ when $x = 1$, and $u = \sqrt{1-0} = 1$ when $x = 0$. So, the correct u-variable form of this integral is

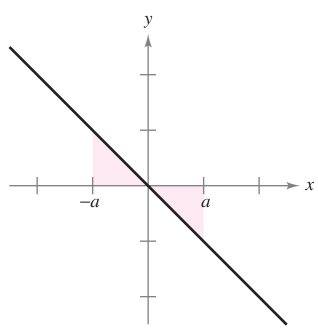
$$-2\int_1^0 (1-u^2)^2 u^2\,du.$$

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral (over an interval that is symmetric about the y-axis or about the origin) by recognizing the integrand to be an even or odd function (see Figure 4.39).



Even function



Odd function
Figure 4.39

THEOREM 4.15 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0.$

Proof Because f is even, you know that $f(x) = f(-x)$. Using Theorem 4.12 with the substitution $u = -x$ produces

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Finally, using Theorem 4.6, you obtain

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

This proves the first property. The proof of the second property is left to you (see Exercise 133).

EXAMPLE 10 Integration of an Odd Function

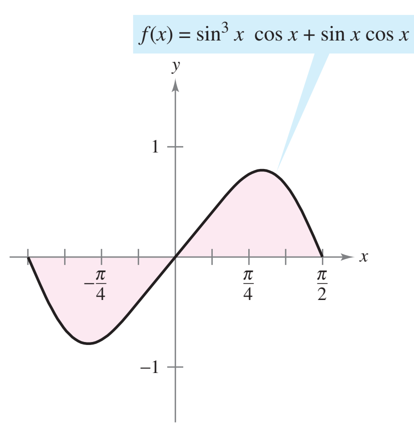
Evaluate $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx.$

Solution Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$\begin{aligned}f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x)\\ &= -\sin^3 x \cos x - \sin x \cos x = -f(x).\end{aligned}$$

So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$, you can apply Theorem 4.15 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx = 0.$$



Because f is an odd function,

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$$

Figure 4.40

NOTE From Figure 4.40 you can see that the two regions on either side of the y-axis have the same area. However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about this in Section 7.1.)

Exercises for Section 4.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, complete the table by identifying u and du for the integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (5x^2+1)^2(10x)\,dx$		
2. $\int x^2\sqrt{x^3+1}\,dx$		
3. $\int \frac{x}{\sqrt{x^2+1}}\,dx$		
4. $\int \sec 2x \tan 2x\,dx$		
5. $\int \tan^2 x \sec^2 x\,dx$		
6. $\int \frac{\cos x}{\sin^2 x}\,dx$		

In Exercises 7–34, find the indefinite integral and check the result by differentiation.

7. $\int (1+2x)^4(2)\,dx$
8. $\int (x^2-9)^3(2x)\,dx$
9. $\int \sqrt{9-x^2}\,(-2x)\,dx$
10. $\int \sqrt[3]{(1-2x^2)}(-4x)\,dx$
11. $\int x^3(x^4+3)^2\,dx$
12. $\int x^2(x^3+5)^4\,dx$
13. $\int x^2(x^3-1)^4\,dx$
14. $\int x(4x^2+3)^3\,dx$
15. $\int t\sqrt{t^2+2}\,dt$
16. $\int t^3\sqrt{t^4+5}\,dt$
17. $\int 5x\sqrt[3]{1-x^2}\,dx$
18. $\int u^2\sqrt{u^3+2}\,du$
19. $\int \frac{x}{(1-x^2)^3}\,dx$
20. $\int \frac{x^3}{(1+x^4)^2}\,dx$
21. $\int \frac{x^2}{(1+x^3)^2}\,dx$
22. $\int \frac{x^2}{(16-x^3)^2}\,dx$
23. $\int \frac{x}{\sqrt{1-x^2}}\,dx$
24. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$
25. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$
26. $\int \left[x^2+\frac{1}{(3x)^2}\right]dx$
27. $\int \frac{1}{\sqrt{2x}}\,dx$
28. $\int \frac{1}{2\sqrt{x}}\,dx$
29. $\int \frac{x^2+3x+7}{\sqrt{x}}\,dx$
30. $\int \frac{t+2t^2}{\sqrt{t}}\,dt$
31. $\int t^2\left(t-\frac{2}{t}\right)dt$
32. $\int \left(\frac{t^3}{3}+\frac{1}{4t^2}\right)dt$
33. $\int (9-y)\sqrt{y}\,dy$
34. $\int 2\pi y(8-y^{3/2})\,dy$

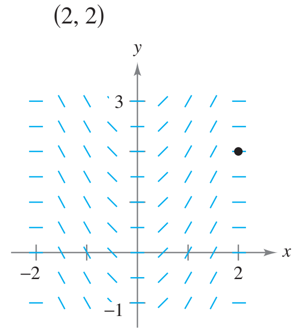
In Exercises 35–38, solve the differential equation.

35. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$
36. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$
37. $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$
38. $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

Slope Fields **In Exercises 39–42, a differential equation, a point, and a slope field are given. A *slope field* consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

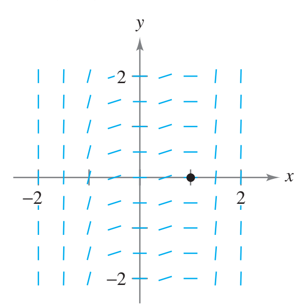
39. $\frac{dy}{dx} = x\sqrt{4-x^2}$

$(2, 2)$



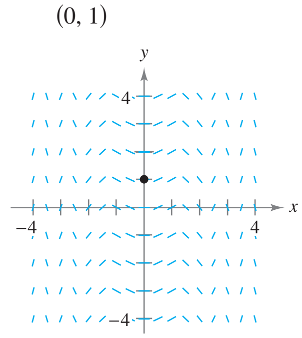
40. $\frac{dy}{dx} = x^2(x^3-1)^2$

$(1, 0)$



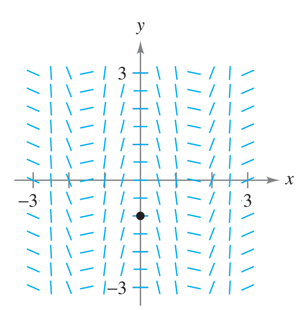
41. $\frac{dy}{dx} = x\cos x^2$

$(0, 1)$



42. $\frac{dy}{dx} = -2\sec(2x)\tan(2x)$

$(0, -1)$



In Exercises 43–56, find the indefinite integral.

43. $\int \pi \sin \pi x \, dx$

44. $\int 4x^3 \sin x^4 \, dx$

45. $\int \sin 2x \, dx$

46. $\int \cos 6x \, dx$

47. $\int \frac{1}{\theta^2} \cos \frac{1}{\theta} \, d\theta$

48. $\int x \sin x^2 \, dx$

49. $\int \sin 2x \cos 2x \, dx$

50. $\int \sec(1 - x) \tan(1 - x) \, dx$

51. $\int \tan^4 x \sec^2 x \, dx$

52. $\int \sqrt{\tan x} \sec^2 x \, dx$

53. $\int \frac{\csc^2 x}{\cot^3 x} \, dx$

54. $\int \frac{\sin x}{\cos^3 x} \, dx$

55. $\int \cot^2 x \, dx$

56. $\int \csc^2\left(\frac{x}{2}\right) dx$

In Exercises 57–62, find an equation for the function *f* that has the given derivative and whose graph passes through the given point.

	Derivative	*Point*
57.	$f'(x) = \cos \frac{x}{2}$	$(0, 3)$
58.	$f'(x) = \pi \sec \pi x \tan \pi x$	$\left(\frac{1}{3}, 1\right)$
59.	$f'(x) = \sin 4x$	$\left(\frac{\pi}{4}, -\frac{3}{4}\right)$
60.	$f'(x) = \sec^2(2x)$	$\left(\frac{\pi}{2}, 2\right)$
61.	$f'(x) = 2x(4x^2 - 10)^2$	$(2, 10)$
62.	$f'(x) = -2x\sqrt{8 - x^2}$	$(2, 7)$

In Exercises 63–70, find the indefinite integral by the method shown in Example 5.

63. $\int x\sqrt{x + 2} \, dx$

64. $\int x\sqrt{2x + 1} \, dx$

65. $\int x^2\sqrt{1 - x} \, dx$

66. $\int (x + 1)\sqrt{2 - x} \, dx$

67. $\int \frac{x^2 - 1}{\sqrt{2x - 1}} \, dx$

68. $\int \frac{2x + 1}{\sqrt{x + 4}} \, dx$

69. $\int \frac{-x}{(x + 1) - \sqrt{x + 1}} \, dx$

70. $\int t \sqrt[3]{t - 4} \, dt$

In Exercises 71–82, evaluate the definite integral. Use a graphing utility to verify your result.

71. $\int_{-1}^{1} x(x^2 + 1)^3 \, dx$

72. $\int_{-2}^{4} x^2(x^3 + 8)^2 \, dx$

73. $\int_{1}^{2} 2x^2\sqrt{x^3 + 1} \, dx$

74. $\int_{0}^{1} x\sqrt{1 - x^2} \, dx$

75. $\int_{0}^{4} \frac{1}{\sqrt{2x + 1}} \, dx$

76. $\int_{0}^{2} \frac{x}{\sqrt{1 + 2x^2}} \, dx$

77. $\int_{1}^{9} \frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)^2} \, dx$

78. $\int_{0}^{2} x \sqrt[3]{4 + x^2} \, dx$

79. $\int_{1}^{2} (x - 1)\sqrt{2 - x} \, dx$

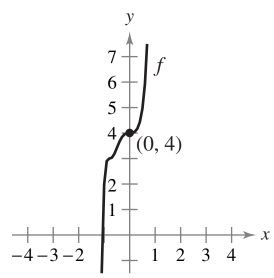
80. $\int_{1}^{5} \frac{x}{\sqrt{2x - 1}} \, dx$

81. $\int_{0}^{\pi/2} \cos\left(\frac{2x}{3}\right) dx$

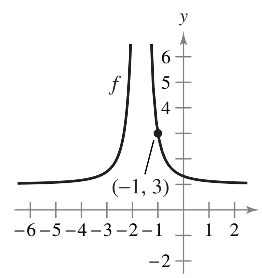
82. $\int_{\pi/3}^{\pi/2} (x + \cos x) \, dx$

Differential Equations **In Exercises 83–86, the graph of a function *f* is shown. Use the differential equation and the given point to find an equation of the function.**

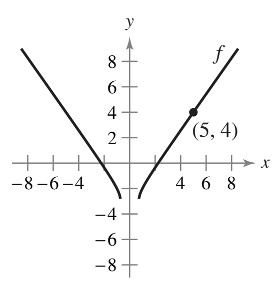
83. $\frac{dy}{dx} = 18x^2(2x^3 + 1)^2$



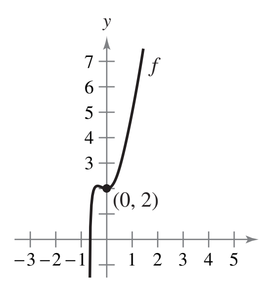
84. $\frac{dy}{dx} = \frac{-48}{(3x + 5)^3}$



85. $\frac{dy}{dx} = \frac{2x}{\sqrt{2x^2 - 1}}$

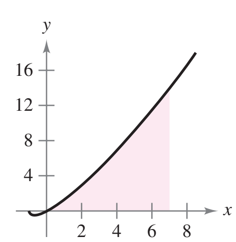


86. $\frac{dy}{dx} = 4x + \frac{9x^2}{(3x^3 + 1)^{(3/2)}}$

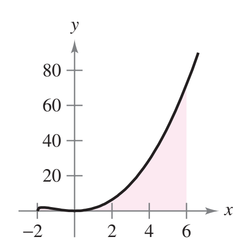


In Exercises 87–92, find the area of the region. Use a graphing utility to verify your result.

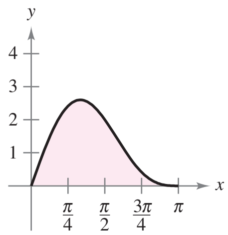
87. $\int_{0}^{7} x \sqrt[3]{x + 1} \, dx$



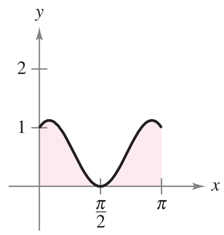
88. $\int_{-2}^{6} x^2 \sqrt[3]{x + 2} \, dx$



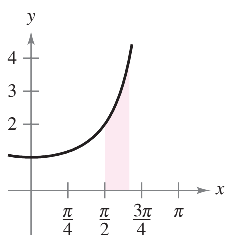
89. $y = 2\sin x + \sin 2x$



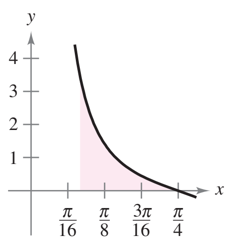
90. $y = \sin x + \cos 2x$



91. $\displaystyle\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx$



92. $\displaystyle\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\, dx$



In Exercises 93–98, use a graphing utility to evaluate the integral. Graph the region whose area is given by the definite integral.

93. $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\, dx$ **94.** $\displaystyle\int_0^2 x^3\sqrt{x+2}\, dx$

95. $\displaystyle\int_3^7 x\sqrt{x-3}\, dx$ **96.** $\displaystyle\int_1^5 x^2\sqrt{x-1}\, dx$

97. $\displaystyle\int_0^3 \left(\theta + \cos\frac{\theta}{6}\right) d\theta$ **98.** $\displaystyle\int_0^{\pi/2} \sin 2x\, dx$

Writing **In Exercises 99 and 100, find the indefinite integral in two ways. Explain any difference in the forms of the answers.**

99. $\displaystyle\int (2x-1)^2\, dx$

100. $\displaystyle\int \sin x \cos x\, dx$

In Exercises 101–104, evaluate the integral using the properties of even and odd functions as an aid.

101. $\displaystyle\int_{-2}^{2} x^2(x^2+1)\, dx$ **102.** $\displaystyle\int_{-2}^{2} x(x^2+1)^3\, dx$

103. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$ **104.** $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx$

105. Use $\int_0^2 x^2\, dx = \frac{8}{3}$ to evaluate each definite integral without using the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_{-2}^{0} x^2\, dx$ (b) $\displaystyle\int_{-2}^{2} x^2\, dx$

(c) $\displaystyle\int_{0}^{2} -x^2\, dx$ (d) $\displaystyle\int_{-2}^{0} 3x^2\, dx$

106. Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x\, dx$ (b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx$

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\, dx$ (d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx$

In Exercises 107 and 108, write the integral as the sum of the integral of an odd function and the integral of an even function. Use this simplification to evaluate the integral.

107. $\displaystyle\int_{-4}^{4} (x^3 + 6x^2 - 2x - 3)\, dx$ **108.** $\displaystyle\int_{-\pi}^{\pi} (\sin 3x + \cos 3x)\, dx$

Writing About Concepts

109. Describe why

$$\int x(5-x^2)^3\, dx \neq \int u^3\, du$$

where $u = 5 - x^2$.

110. Without integrating, explain why

$$\int_{-2}^{2} x(x^2+1)^2\, dx = 0.$$

111. ***Cash Flow*** The rate of disbursement dQ/dt of a 2 million dollar federal grant is proportional to the square of $100 - t$. Time t is measured in days $(0 \le t \le 100)$, and Q is the amount that remains to be disbursed. Find the amount that remains to be disbursed after 50 days. Assume that all the money will be disbursed in 100 days.

112. ***Depreciation*** The rate of depreciation dV/dt of a machine is inversely proportional to the square of $t + 1$, where V is the value of the machine t years after it was purchased. The initial value of the machine was \$500,000, and its value decreased \$100,000 in the first year. Estimate its value after 4 years.

113. ***Rainfall*** The normal monthly rainfall at the Seattle-Tacoma airport can be approximated by the model

$$R = 3.121 + 2.399\sin(0.524t + 1.377)$$

where R is measured in inches and t is the time in months, with $t = 1$ corresponding to January. (*Source: U.S. National Oceanic and Atmospheric Administration*)

(a) Determine the extrema of the function over a one-year period.

(b) Use integration to approximate the normal annual rainfall. (*Hint:* Integrate over the interval $[0, 12]$.)

(c) Approximate the average monthly rainfall during the months of October, November, and December.

114. ***Sales*** The sales S (in thousands of units) of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time in months, with $t = 1$ corresponding to January. Find the average sales for each time period.

(a) The first quarter $(0 \le t \le 3)$

(b) The second quarter $(3 \le t \le 6)$

(c) The entire year $(0 \le t \le 12)$

115. ***Water Supply*** A model for the flow rate of water at a pumping station on a given day is

$$R(t) = 53 + 7 \sin\left(\frac{\pi t}{6} + 3.6\right) + 9 \cos\left(\frac{\pi t}{12} + 8.9\right)$$

where $0 \le t \le 24$. R is the flow rate in thousands of gallons per hour, and t is the time in hours.

(a) Use a graphing utility to graph the rate function and approximate the maximum flow rate at the pumping station.

(b) Approximate the total volume of water pumped in 1 day.

116. ***Electricity*** The oscillating current in an electrical circuit is

$$I = 2 \sin(60\pi t) + \cos(120\pi t)$$

where I is measured in amperes and t is measured in seconds. Find the average current for each time interval.

(a) $0 \le t \le \frac{1}{60}$

(b) $0 \le t \le \frac{1}{240}$

(c) $0 \le t \le \frac{1}{30}$

Probability **In Exercises 117 and 118, the function**

$$f(x) = kx^n(1 - x)^m, \quad 0 \le x \le 1$$

where $n > 0$, $m > 0$, and k is a constant, can be used to represent various probability distributions. If k is chosen such that

$$\int_0^1 f(x)\,dx = 1$$

the probability that x will fall between a and b $(0 \le a \le b \le 1)$ is

$$P_{a,b} = \int_a^b f(x)\,dx.$$

117. The probability that a person will remember between $a\%$ and $b\%$ of material learned in an experiment is

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1 - x}\,dx$$

where x represents the percent remembered. (See figure.)

(a) For a randomly chosen individual, what is the probability that he or she will recall between 50% and 75% of the material?

(b) What is the median percent recall? That is, for what value of b is it true that the probability of recalling 0 to b is 0.5?

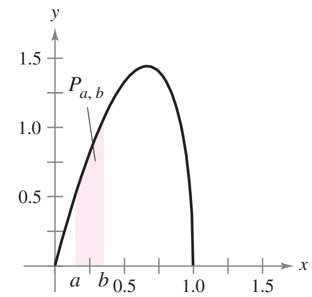


Figure for 117

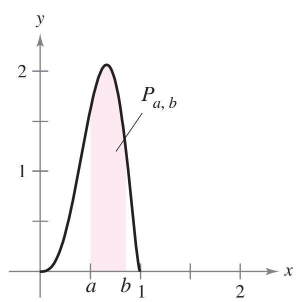
118. The probability that ore samples taken from a region contain between $a\%$ and $b\%$ iron is

$$P_{a,b} = \int_a^b \frac{1155}{32}x^3(1 - x)^{3/2}\,dx$$

where x represents the percent of iron. (See figure.) What is the probability that a sample will contain between

(a) 0% and 25% iron?

(b) 50% and 100% iron?



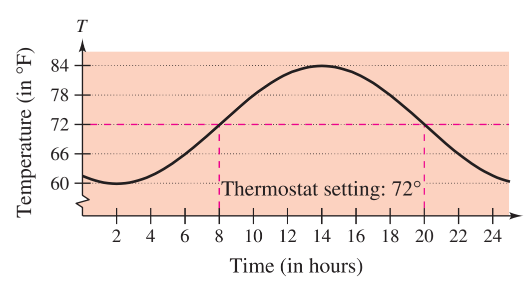
119. ***Temperature*** The temperature in degrees Fahrenheit in a house is

$$T = 72 + 12 \sin\left[\frac{\pi(t - 8)}{12}\right]$$

where t is time in hours, with $t = 0$ representing midnight. The hourly cost of cooling a house is \$0.10 per degree.

(a) Find the cost C of cooling the house if its thermostat is set at 72°F by evaluating the integral

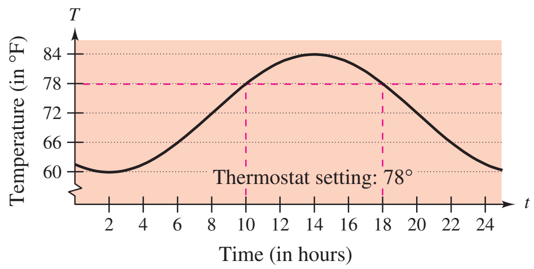
$$C = 0.1\int_8^{20} \left[72 + 12 \sin \frac{\pi(t - 8)}{12} - 72\right] dt. \text{ (See figure.)}$$



(b) Find the savings from resetting the thermostat to 78°F by evaluating the integral

$$C = 0.1\int_{10}^{18}\left[72 + 12\sin\frac{\pi(t-8)}{12} - 78\right]dt.$$

(See figure.)



120. ***Manufacturing*** A manufacturer of fertilizer finds that national sales of fertilizer follow the seasonal pattern

$$F = 100{,}000\left[1 + \sin\frac{2\pi(t-60)}{365}\right]$$

where F is measured in pounds and t represents the time in days, with $t = 1$ corresponding to January 1. The manufacturer wants to set up a schedule to produce a uniform amount of fertilizer each day. What should this amount be?

121. ***Graphical Analysis*** Consider the functions f and g, where

$$f(x) = 6\sin x\cos^2 x \quad \text{and} \quad g(t) = \int_0^t f(x)\,dx.$$

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Explain why g is nonnegative.

(c) Identify the points on the graph of g that correspond to the extrema of f.

(d) Does each of the zeros of f correspond to an extremum of g? Explain.

(e) Consider the function

$$h(t) = \int_{\pi/2}^{t} f(x)\,dx.$$

Use a graphing utility to graph h. What is the relationship between g and h? Verify your conjecture.

122. Find $\displaystyle\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{\sin(i\pi/n)}{n}$ by evaluating an appropriate definite integral over the interval $[0, 1]$.

123. (a) Show that $\int_0^1 x^2(1-x)^5\,dx = \int_0^1 x^5(1-x)^2\,dx$.

(b) Show that $\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$.

124. (a) Show that $\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx$.

(b) Show that $\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx$, where n is a positive integer.

True or False? **In Exercises 125–130, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

125. $\displaystyle\int (2x+1)^2\,dx = \tfrac{1}{3}(2x+1)^3 + C$

126. $\displaystyle\int x(x^2+1)\,dx = \tfrac{1}{2}x^2\left(\tfrac{1}{3}x^3 + x\right) + C$

127. $\displaystyle\int_{-10}^{10}(ax^3 + bx^2 + cx + d)\,dx = 2\int_0^{10}(bx^2 + d)\,dx$

128. $\displaystyle\int_a^b \sin x\,dx = \int_a^{b+2\pi}\sin x\,dx$

129. $\displaystyle 4\int \sin x\cos x\,dx = -\cos 2x + C$

130. $\displaystyle\int \sin^2 2x\cos 2x\,dx = \tfrac{1}{3}\sin^3 2x + C$

131. Assume that f is continuous everywhere and that c is a constant. Show that

$$\int_{ca}^{cb} f(x)\,dx = c\int_a^b f(cx)\,dx.$$

132. (a) Verify that $\sin u - u\cos u + C = \int u\sin u\,du$.

(b) Use part (a) to show that $\int_0^{\pi^2}\sin\sqrt{x}\,dx = 2\pi$.

133. Complete the proof of Theorem 4.15.

134. Show that if f is continuous on the entire real number line, then

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(x)\,dx.$$

Putnam Exam Challenge

135. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

show that the equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ has at least one real zero.

136. Find all the continuous positive functions $f(x)$, for $0 \le x \le 1$, such that

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 f(x)x\,dx = \alpha$$

$$\int_0^1 f(x)x^2\,dx = \alpha^2$$

where α is a real number.

These problems were composed by the Committee on the Putnam Prize Competition.