5.1 Using Fundamental Identities

What you should learn

- Recognize and write the fundamental trigonometric identities.
- Use the fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions.

Why you should learn it

Fundamental trigonometric identities can be used to simplify trigonometric expressions. For instance, in Exercise 99 on page 381, you can use trigonometric identities to simplify an expression for the coefficient of friction.

Introduction

In Chapter 4, you studied the basic definitions, properties, graphs, and applications of the individual trigonometric functions. In this chapter, you will learn how to use the fundamental identities to do the following.

1. Evaluate trigonometric functions.
2. Simplify trigonometric expressions.
3. Develop additional trigonometric identities.
4. Solve trigonometric equations.

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u}$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1 \qquad 1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \csc^2 u$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

$$\sec\left(\frac{\pi}{2} - u\right) = \csc u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Even/Odd Identities

$$\sin(-u) = -\sin u \qquad \cos(-u) = \cos u \qquad \tan(-u) = -\tan u$$

$$\csc(-u) = -\csc u \qquad \sec(-u) = \sec u \qquad \cot(-u) = -\cot u$$

Pythagorean identities are sometimes used in radical form such as

$$\sin u = \pm\sqrt{1 - \cos^2 u}$$

or

$$\tan u = \pm\sqrt{\sec^2 u - 1}$$

where the sign depends on the choice of u.

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain additional resources related to the concepts discussed in this chapter.

STUDY TIP

You should learn the fundamental trigonometric identities well, because they are used frequently in trigonometry and they will also appear later in calculus. Note that u can be an angle, a real number, or a variable.

Using the Fundamental Identities

One common use of trigonometric identities is to use given values of trigonometric functions to evaluate other trigonometric functions.

Example 1 Using Identities to Evaluate a Function

Use the values $\sec u = -\frac{3}{2}$ and $\tan u > 0$ to find the values of all six trigonometric functions.

Solution

Using a reciprocal identity, you have

$$\cos u = \frac{1}{\sec u} = \frac{1}{-3/2} = -\frac{2}{3}.$$

Using a Pythagorean identity, you have

$$\sin^2 u = 1 - \cos^2 u \qquad \text{Pythagorean identity}$$

$$= 1 - \left(-\frac{2}{3}\right)^2 \qquad \text{Substitute } -\tfrac{2}{3} \text{ for } \cos u.$$

$$= 1 - \frac{4}{9} = \frac{5}{9}. \qquad \text{Simplify.}$$

Because $\sec u < 0$ and $\tan u > 0$, it follows that u lies in Quadrant III. Moreover, because $\sin u$ is negative when u is in Quadrant III, you can choose the negative root and obtain $\sin u = -\sqrt{5}/3$. Now, knowing the values of the sine and cosine, you can find the values of all six trigonometric functions.

$$\sin u = -\frac{\sqrt{5}}{3} \qquad \csc u = \frac{1}{\sin u} = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\cos u = -\frac{2}{3} \qquad \sec u = \frac{1}{\cos u} = -\frac{3}{2}$$

$$\tan u = \frac{\sin u}{\cos u} = \frac{-\sqrt{5}/3}{-2/3} = \frac{\sqrt{5}}{2} \qquad \cot u = \frac{1}{\tan u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

CHECKPOINT Now try Exercise 11.

Example 2 Simplifying a Trigonometric Expression

Simplify $\sin x \cos^2 x - \sin x$.

Solution

First factor out a common monomial factor and then use a fundamental identity.

$$\sin x \cos^2 x - \sin x = \sin x(\cos^2 x - 1) \qquad \text{Factor out common monomial factor.}$$

$$= -\sin x(1 - \cos^2 x) \qquad \text{Factor out } -1.$$

$$= -\sin x(\sin^2 x) \qquad \text{Pythagorean identity}$$

$$= -\sin^3 x \qquad \text{Multiply.}$$

CHECKPOINT Now try Exercise 45.

Technology

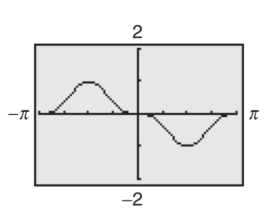
You can use a graphing utility to check the result of Example 2. To do this, graph

$$y_1 = \sin x \cos^2 x - \sin x$$

and

$$y_2 = -\sin^3 x$$

in the same viewing window, as shown below. Because Example 2 shows the equivalence algebraically and the two graphs appear to coincide, you can conclude that the expressions are equivalent.



Remind students that they must use an algebraic approach to prove that two expressions are equivalent. A graphical approach can only confirm that the simplification found using algebraic techniques is correct.

When factoring trigonometric expressions, it is helpful to find a special polynomial factoring form that fits the expression, as shown in Example 3.

Example 3 Factoring Trigonometric Expressions

Factor each expression.

a. $\sec^2\theta - 1$ **b.** $4\tan^2\theta + \tan\theta - 3$

Solution

a. Here you have the difference of two squares, which factors as

$$\sec^2\theta - 1 = (\sec\theta - 1)(\sec\theta + 1).$$

b. This expression has the polynomial form $ax^2 + bx + c$, and it factors as

$$4\tan^2\theta + \tan\theta - 3 = (4\tan\theta - 3)(\tan\theta + 1).$$

CHECKPOINT Now try Exercise 47.

On occasion, factoring or simplifying can best be done by first rewriting the expression in terms of just *one* trigonometric function or in terms of *sine and cosine only*. These strategies are illustrated in Examples 4 and 5, respectively.

Example 4 Factoring a Trigonometric Expression

Factor $\csc^2 x - \cot x - 3$.

Solution

Use the identity $\csc^2 x = 1 + \cot^2 x$ to rewrite the expression in terms of the cotangent.

$$\csc^2 x - \cot x - 3 = (1 + \cot^2 x) - \cot x - 3 \quad \text{Pythagorean identity}$$

$$= \cot^2 x - \cot x - 2 \quad \text{Combine like terms.}$$

$$= (\cot x - 2)(\cot x + 1) \quad \text{Factor.}$$

CHECKPOINT Now try Exercise 51.

Example 5 Simplifying a Trigonometric Expression

Simplify $\sin t + \cot t \cos t$.

Solution

Begin by rewriting $\cot t$ in terms of sine and cosine.

$$\sin t + \cot t \cos t = \sin t + \left(\frac{\cos t}{\sin t}\right)\cos t \quad \text{Quotient identity}$$

$$= \frac{\sin^2 t + \cos^2 t}{\sin t} \quad \text{Add fractions.}$$

$$= \frac{1}{\sin t} \quad \text{Pythagorean identity}$$

$$= \csc t \quad \text{Reciprocal identity}$$

CHECKPOINT Now try Exercise 57.

STUDY TIP

Remember that when adding rational expressions, you must first find the least common denominator (LCD). In Example 5, the LCD is $\sin t$.

Example 6 Adding Trigonometric Expressions

Perform the addition and simplify.

$$\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}$$

Solution

$$\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{(\sin\theta)(\sin\theta)+(\cos\theta)(1+\cos\theta)}{(1+\cos\theta)(\sin\theta)}$$

$$=\frac{\sin^2\theta+\cos^2\theta+\cos\theta}{(1+\cos\theta)(\sin\theta)} \quad \text{Multiply.}$$

$$=\frac{\cancel{1+\cos\theta}}{\cancel{(1+\cos\theta)}(\sin\theta)} \quad \text{Pythagorean identity: } \sin^2\theta+\cos^2\theta=1$$

$$=\frac{1}{\sin\theta} \quad \text{Divide out common factor.}$$

$$=\csc\theta \quad \text{Reciprocal identity}$$

CHECKPOINT Now try Exercise 61.

The last two examples in this section involve techniques for rewriting expressions in forms that are used in calculus.

Example 7 Rewriting a Trigonometric Expression

Rewrite $\dfrac{1}{1+\sin x}$ so that it is *not* in fractional form.

Solution

From the Pythagorean identity $\cos^2 x = 1-\sin^2 x = (1-\sin x)(1+\sin x)$, you can see that multiplying both the numerator and the denominator by $(1-\sin x)$ will produce a monomial denominator.

$$\frac{1}{1+\sin x}=\frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x} \quad \text{Multiply numerator and denominator by } (1-\sin x).$$

$$=\frac{1-\sin x}{1-\sin^2 x} \quad \text{Multiply.}$$

$$=\frac{1-\sin x}{\cos^2 x} \quad \text{Pythagorean identity}$$

$$=\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x} \quad \text{Write as separate fractions.}$$

$$=\frac{1}{\cos^2 x}-\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} \quad \text{Product of fractions}$$

$$=\sec^2 x-\tan x\sec x \quad \text{Reciprocal and quotient identities}$$

CHECKPOINT Now try Exercise 65.

Example 8 Trigonometric Substitution

Use the substitution $x = 2\tan\theta,\ 0 < \theta < \pi/2$, to write

$$\sqrt{4 + x^2}$$

as a trigonometric function of θ.

Solution

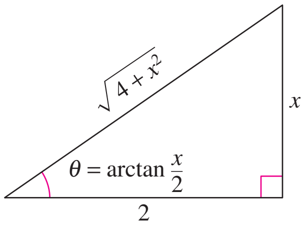
Begin by letting $x = 2\tan\theta$. Then, you can obtain

$$\sqrt{4 + x^2} = \sqrt{4 + (2\tan\theta)^2} \quad \text{Substitute } 2\tan\theta \text{ for } x.$$
$$= \sqrt{4 + 4\tan^2\theta} \quad \text{Rule of exponents}$$
$$= \sqrt{4(1 + \tan^2\theta)} \quad \text{Factor.}$$
$$= \sqrt{4\sec^2\theta} \quad \text{Pythagorean identity}$$
$$= 2\sec\theta. \quad \sec\theta > 0 \text{ for } 0 < \theta < \pi/2$$

CHECKPOINT Now try Exercise 77.

Activities

1. Simplify, using the fundamental trigonometric identities.
   $\dfrac{\cot^2\theta}{\csc^2\theta}$
   Answer: $\cos^2\theta$
2. Use the trigonometric substitution $x = 4\sec\theta$ to rewrite the expression $\sqrt{x^2 - 16}$ as a trigonometric function of θ, where $0 < \theta < \dfrac{\pi}{2}$.
   Answer: $4\tan\theta$



Angle whose tangent is $\frac{x}{2}$.

FIGURE 5.1

Figure 5.1 shows the right triangle illustration of the trigonometric substitution $x = 2\tan\theta$ in Example 8. You can use this triangle to check the solution of Example 8. For $0 < \theta < \pi/2$, you have

$$\text{opp} = x, \quad \text{adj} = 2, \quad \text{and} \quad \text{hyp} = \sqrt{4 + x^2}.$$

With these expressions, you can write the following.

$$\sec\theta = \frac{\text{hyp}}{\text{adj}}$$
$$\sec\theta = \frac{\sqrt{4 + x^2}}{2}$$
$$2\sec\theta = \sqrt{4 + x^2}$$

So, the solution checks.

Example 9 Rewriting a Logarithmic Expression

Rewrite $\ln|\csc\theta| + \ln|\tan\theta|$ as a single logarithm and simplify the result.

Solution

$$\ln|\csc\theta| + \ln|\tan\theta| = \ln|\csc\theta\tan\theta| \quad \text{Product Property of Logarithms}$$
$$= \ln\left|\frac{1}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}\right| \quad \text{Reciprocal and quotient identities}$$
$$= \ln\left|\frac{1}{\cos\theta}\right| \quad \text{Simplify.}$$
$$= \ln|\sec\theta| \quad \text{Reciprocal identity}$$

CHECKPOINT Now try Exercise 91.

5.1 Exercises

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

VOCABULARY CHECK: Fill in the blank to complete the trigonometric identity.

1. $\dfrac{\sin u}{\cos u} = $ ________
2. $\dfrac{1}{\sec u} = $ ________
3. $\dfrac{1}{\tan u} = $ ________
4. $\dfrac{1}{\sin u} = $ ________
5. $1 + $ ________ $= \csc^2 u$
6. $1 + \tan^2 u = $ ________
7. $\sin\left(\dfrac{\pi}{2} - u\right) = $ ________
8. $\sec\left(\dfrac{\pi}{2} - u\right) = $ ________
9. $\cos(-u) = $ ________
10. $\tan(-u) = $ ________

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–14, use the given values to evaluate (if possible) all six trigonometric functions.

1. $\sin x = \dfrac{\sqrt{3}}{2}, \quad \cos x = -\dfrac{1}{2}$
2. $\tan x = \dfrac{\sqrt{3}}{3}, \quad \cos x = -\dfrac{\sqrt{3}}{2}$
3. $\sec \theta = \sqrt{2}, \quad \sin \theta = -\dfrac{\sqrt{2}}{2}$
4. $\csc \theta = \frac{5}{3}, \quad \tan \theta = \frac{3}{4}$
5. $\tan x = \frac{5}{12}, \quad \sec x = -\frac{13}{12}$
6. $\cot \phi = -3, \quad \sin \phi = \dfrac{\sqrt{10}}{10}$
7. $\sec \phi = \dfrac{3}{2}, \quad \csc \phi = -\dfrac{3\sqrt{5}}{5}$
8. $\cos\left(\dfrac{\pi}{2} - x\right) = \dfrac{3}{5}, \quad \cos x = \dfrac{4}{5}$
9. $\sin(-x) = -\dfrac{1}{3}, \quad \tan x = -\dfrac{\sqrt{2}}{4}$
10. $\sec x = 4, \quad \sin x > 0$
11. $\tan \theta = 2, \quad \sin \theta < 0$
12. $\csc \theta = -5, \quad \cos \theta < 0$
13. $\sin \theta = -1, \quad \cot \theta = 0$
14. $\tan \theta$ is undefined, $\quad \sin \theta > 0$

In Exercises 15–20, match the trigonometric expression with one of the following.

(a) $\sec x$ (b) -1 (c) $\cot x$
(d) 1 (e) $-\tan x$ (f) $\sin x$

15. $\sec x \cos x$
16. $\tan x \csc x$
17. $\cot^2 x - \csc^2 x$
18. $(1 - \cos^2 x)(\csc x)$
19. $\dfrac{\sin(-x)}{\cos(-x)}$
20. $\dfrac{\sin[(\pi/2) - x]}{\cos[(\pi/2) - x]}$

In Exercises 21–26, match the trigonometric expression with one of the following.

(a) $\csc x$ (b) $\tan x$ (c) $\sin^2 x$
(d) $\sin x \tan x$ (e) $\sec^2 x$ (f) $\sec^2 x + \tan^2 x$

21. $\sin x \sec x$
22. $\cos^2 x(\sec^2 x - 1)$
23. $\sec^4 x - \tan^4 x$
24. $\cot x \sec x$
25. $\dfrac{\sec^2 x - 1}{\sin^2 x}$
26. $\dfrac{\cos^2[(\pi/2) - x]}{\cos x}$

In Exercises 27–44, use the fundamental identities to simplify the expression. There is more than one correct form of each answer.

27. $\cot \theta \sec \theta$
28. $\cos \beta \tan \beta$
29. $\sin \phi(\csc \phi - \sin \phi)$
30. $\sec^2 x(1 - \sin^2 x)$
31. $\dfrac{\cot x}{\csc x}$
32. $\dfrac{\csc \theta}{\sec \theta}$
33. $\dfrac{1 - \sin^2 x}{\csc^2 x - 1}$
34. $\dfrac{1}{\tan^2 x + 1}$
35. $\sec \alpha \cdot \dfrac{\sin \alpha}{\tan \alpha}$
36. $\dfrac{\tan^2 \theta}{\sec^2 \theta}$
37. $\cos\left(\dfrac{\pi}{2} - x\right)\sec x$
38. $\cot\left(\dfrac{\pi}{2} - x\right)\cos x$
39. $\dfrac{\cos^2 y}{1 - \sin y}$
40. $\cos t(1 + \tan^2 t)$
41. $\sin \beta \tan \beta + \cos \beta$
42. $\csc \phi \tan \phi + \sec \phi$
43. $\cot u \sin u + \tan u \cos u$
44. $\sin \theta \sec \theta + \cos \theta \csc \theta$

In Exercises 45–56, factor the expression and use the fundamental identities to simplify. There is more than one correct form of each answer.

45. $\tan^2 x - \tan^2 x \sin^2 x$

46. $\sin^2 x \csc^2 x - \sin^2 x$

47. $\sin^2 x \sec^2 x - \sin^2 x$

48. $\cos^2 x + \cos^2 x \tan^2 x$

49. $\dfrac{\sec^2 x - 1}{\sec x - 1}$

50. $\dfrac{\cos^2 x - 4}{\cos x - 2}$

51. $\tan^4 x + 2\tan^2 x + 1$

52. $1 - 2\cos^2 x + \cos^4 x$

53. $\sin^4 x - \cos^4 x$

54. $\sec^4 x - \tan^4 x$

55. $\csc^3 x - \csc^2 x - \csc x + 1$

56. $\sec^3 x - \sec^2 x - \sec x + 1$

In Exercises 57–60, perform the multiplication and use the fundamental identities to simplify. There is more than one correct form of each answer.

57. $(\sin x + \cos x)^2$

58. $(\cot x + \csc x)(\cot x - \csc x)$

59. $(2\csc x + 2)(2\csc x - 2)$

60. $(3 - 3\sin x)(3 + 3\sin x)$

In Exercises 61–64, perform the addition or subtraction and use the fundamental identities to simplify. There is more than one correct form of each answer.

61. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x}$

62. $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1}$

63. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x}$

64. $\tan x - \dfrac{\sec^2 x}{\tan x}$

In Exercises 65–68, rewrite the expression so that it is not in fractional form. There is more than one correct form of each answer.

65. $\dfrac{\sin^2 y}{1 - \cos y}$

66. $\dfrac{5}{\tan x + \sec x}$

67. $\dfrac{3}{\sec x - \tan x}$

68. $\dfrac{\tan^2 x}{\csc x + 1}$

Numerical and Graphical Analysis **In Exercises 69–72, use a graphing utility to complete the table and graph the functions. Make a conjecture about y_1 and y_2.**

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y_1							
y_2							

69. $y_1 = \cos\left(\dfrac{\pi}{2} - x\right), \quad y_2 = \sin x$

70. $y_1 = \sec x - \cos x, \quad y_2 = \sin x \tan x$

71. $y_1 = \dfrac{\cos x}{1 - \sin x}, \quad y_2 = \dfrac{1 + \sin x}{\cos x}$

72. $y_1 = \sec^4 x - \sec^2 x, \quad y_2 = \tan^2 x + \tan^4 x$

In Exercises 73–76, use a graphing utility to determine which of the six trigonometric functions is equal to the expression. Verify your answer algebraically.

73. $\cos x \cot x + \sin x$

74. $\sec x \csc x - \tan x$

75. $\dfrac{1}{\sin x}\left(\dfrac{1}{\cos x} - \cos x\right)$

76. $\dfrac{1}{2}\left(\dfrac{1 + \sin \theta}{\cos \theta} + \dfrac{\cos \theta}{1 + \sin \theta}\right)$

In Exercises 77–82, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.

77. $\sqrt{9 - x^2}, \quad x = 3\cos\theta$

78. $\sqrt{64 - 16x^2}, \quad x = 2\cos\theta$

79. $\sqrt{x^2 - 9}, \quad x = 3\sec\theta$

80. $\sqrt{x^2 - 4}, \quad x = 2\sec\theta$

81. $\sqrt{x^2 + 25}, \quad x = 5\tan\theta$

82. $\sqrt{x^2 + 100}, \quad x = 10\tan\theta$

In Exercises 83–86, use the trigonometric substitution to write the algebraic equation as a trigonometric function of θ, where $-\pi/2 < \theta < \pi/2$. Then find $\sin\theta$ and $\cos\theta$.

83. $3 = \sqrt{9 - x^2}, \quad x = 3\sin\theta$

84. $3 = \sqrt{36 - x^2}, \quad x = 6\sin\theta$

85. $2\sqrt{2} = \sqrt{16 - 4x^2}, \quad x = 2\cos\theta$

86. $-5\sqrt{3} = \sqrt{100 - x^2}, \quad x = 10\cos\theta$

In Exercises 87–90, use a graphing utility to solve the equation for θ, where $0 \le \theta < 2\pi$.

87. $\sin\theta = \sqrt{1 - \cos^2\theta}$

88. $\cos\theta = -\sqrt{1 - \sin^2\theta}$

89. $\sec\theta = \sqrt{1 + \tan^2\theta}$

90. $\csc\theta = \sqrt{1 + \cot^2\theta}$

In Exercises 91–94, rewrite the expression as a single logarithm and simplify the result.

91. $\ln|\cos x| - \ln|\sin x|$

92. $\ln|\sec x| + \ln|\sin x|$

93. $\ln|\cot t| + \ln(1 + \tan^2 t)$

94. $\ln(\cos^2 t) + \ln(1 + \tan^2 t)$

In Exercises 95–98, use a calculator to demonstrate the identity for each value of θ.

95. $\csc^2 \theta - \cot^2 \theta = 1$

(a) $\theta = 132°$, (b) $\theta = \frac{2\pi}{7}$

96. $\tan^2 \theta + 1 = \sec^2 \theta$

(a) $\theta = 346°$, (b) $\theta = 3.1$

97. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$

(a) $\theta = 80°$, (b) $\theta = 0.8$

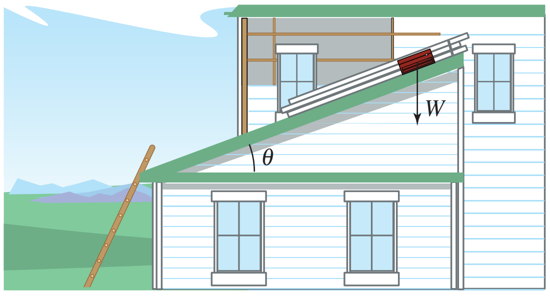
98. $\sin(-\theta) = -\sin \theta$

(a) $\theta = 250°$, (b) $\theta = \frac{1}{2}$

99. ***Friction*** The forces acting on an object weighing W units on an inclined plane positioned at an angle of θ with the horizontal (see figure) are modeled by

$$\mu W \cos \theta = W \sin \theta$$

where μ is the coefficient of friction. Solve the equation for μ and simplify the result.



100. ***Rate of Change*** The rate of change of the function

$$f(x) = -\csc x - \sin x$$

is given by the expression

$$\csc x \cot x - \cos x.$$

Show that this expression can also be written as $\cos x \cot^2 x$.

Synthesis

True or False? **In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.**

101. The even and odd trigonometric identities are helpful for determining whether the value of a trigonometric function is positive or negative.

102. A cofunction identity can be used to transform a tangent function so that it can be represented by a cosecant function.

In Exercises 103–106, fill in the blanks. (*Note:* The notation $x \to c^+$ indicates that x approaches c from the right and $x \to c^-$ indicates that x approaches c from the left.)

103. As $x \to \frac{\pi}{2}^-$, $\sin x \to$ ____ and $\csc x \to$ ____.

104. As $x \to 0^+$, $\cos x \to$ ____ and $\sec x \to$ ____.

105. As $x \to \frac{\pi}{2}^-$, $\tan x \to$ ____ and $\cot x \to$ ____.

106. As $x \to \pi^+$, $\sin x \to$ ____ and $\csc x \to$ ____.

In Exercises 107–112, determine whether or not the equation is an identity, and give a reason for your answer.

107. $\cos \theta = \sqrt{1 - \sin^2 \theta}$

108. $\cot \theta = \sqrt{\csc^2 \theta + 1}$

109. $\frac{(\sin k\theta)}{(\cos k\theta)} = \tan \theta$, k is a constant.

110. $\frac{1}{(5 \cos \theta)} = 5 \sec \theta$

111. $\sin \theta \csc \theta = 1$

112. $\csc^2 \theta = 1$

113. Use the definitions of sine and cosine to derive the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$.

114. ***Writing*** Use the Pythagorean identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

to derive the other Pythagorean identities, $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$. Discuss how to remember these identities and other fundamental identities.

Skills Review

In Exercises 115 and 116, perform the operation and simplify.

115. $(\sqrt{x} + 5)(\sqrt{x} - 5)$

116. $(2\sqrt{z} + 3)^2$

In Exercises 117–120, perform the addition or subtraction and simplify.

117. $\frac{1}{x + 5} + \frac{x}{x - 8}$

118. $\frac{6x}{x - 4} - \frac{3}{4 - x}$

119. $\frac{2x}{x^2 - 4} - \frac{7}{x + 4}$

120. $\frac{x}{x^2 - 25} + \frac{x^2}{x - 5}$

In Exercises 121–124, sketch the graph of the function. (Include two full periods.)

121. $f(x) = \frac{1}{2} \sin \pi x$

122. $f(x) = -2 \tan \frac{\pi x}{2}$

123. $f(x) = \frac{1}{2} \sec\left(x + \frac{\pi}{4}\right)$

124. $f(x) = \frac{3}{2} \cos(x - \pi) + 3$