Section 5.4 Exponential Functions: Differentiation and Integration

- **Develop properties of the natural exponential function.**
- **Differentiate natural exponential functions.**
- **Integrate natural exponential functions.**

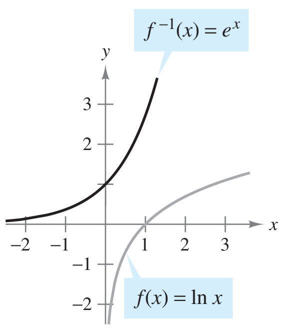
The Natural Exponential Function

The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}. The domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 5.19. So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$

If x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$



The inverse function of the natural logarithmic function is the natural exponential function.
Figure 5.19

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. The following definition extends the meaning of e^x to include *all* real values of x.

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

The Number *e*

The symbol *e* was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

EXAMPLE 1 Solving Exponential Equations

Solve $7 = e^{x+1}$.

Solution You can convert from exponential form to logarithmic form by *taking the natural logarithim of each side* of the equation.

$$7 = e^{x+1} \qquad \text{Write original equation.}$$
$$\ln 7 = \ln(e^{x+1}) \qquad \text{Take natural logarithm of each side.}$$
$$\ln 7 = x + 1 \qquad \text{Apply inverse property.}$$
$$-1 + \ln 7 = x \qquad \text{Solve for } x.$$
$$0.946 \approx x \qquad \text{Use a calculator.}$$

Check this solution in the original equation.

EXAMPLE 2 Solving a Logarithmic Equation

Solve $\ln(2x - 3) = 5$.

Solution To convert from logarithmic form to exponential form, you can *exponentiate each side* of the logarithmic equation.

$\ln(2x - 3) = 5$	Write original equation.
$e^{\ln(2x-3)} = e^5$	Exponentiate each side.
$2x - 3 = e^5$	Apply inverse property.
$x = \frac{1}{2}(e^5 + 3)$	Solve for x.
$x \approx 75.707$	Use a calculator.

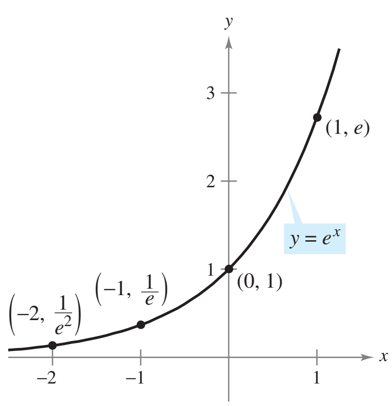
The familiar rules for operating with rational exponents can be extended to the natural exponential function, as shown in the following theorem.

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$

2. $\dfrac{e^a}{e^b} = e^{a-b}$



The natural exponential function is increasing, and its graph is concave upward.
Figure 5.20

Proof To prove Property 1, you can write

$$\ln(e^a e^b) = \ln(e^a) + \ln(e^b)$$
$$= a + b$$
$$= \ln(e^{a+b}).$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$e^a e^b = e^{a+b}.$$

The proof of the second property is left to you (see Exercise 129).

In Section 5.3, you learned that an inverse function f^{-1} shares many properties with f. So, the natural exponential function inherits the following properties from the natural logarithmic function (see Figure 5.20).

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to \infty} e^x = \infty$

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution to the differential equation $y' = y$. This result is stated in the next theorem.

FOR FURTHER INFORMATION To find out about derivatives of exponential functions of order 1/2, see the article "A Child's Garden of Fractional Derivatives" by Marcia Kleinz and Thomas J. Osler in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

THEOREM 5.11 Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[e^x] = e^x$

2. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$

Proof To prove Property 1, use the fact that $\ln e^x = x$, and differentiate each side of the equation.

$$\ln e^x = x \qquad \text{Definition of exponential function}$$

$$\frac{d}{dx}[\ln e^x] = \frac{d}{dx}[x] \qquad \text{Differentiate each side with respect to } x.$$

$$\frac{1}{e^x}\frac{d}{dx}[e^x] = 1$$

$$\frac{d}{dx}[e^x] = e^x$$

The derivative of e^u follows from the Chain Rule.

NOTE You can interpret this theorem geometrically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point.

EXAMPLE 3 Differentiating Exponential Functions

a. $\frac{d}{dx}[e^{2x-1}] = e^u \frac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\frac{d}{dx}[e^{-3/x}] = e^u \frac{du}{dx} = \left(\frac{3}{x^2}\right)e^{-3/x} = \frac{3e^{-3/x}}{x^2}$ $\qquad u = -\frac{3}{x}$

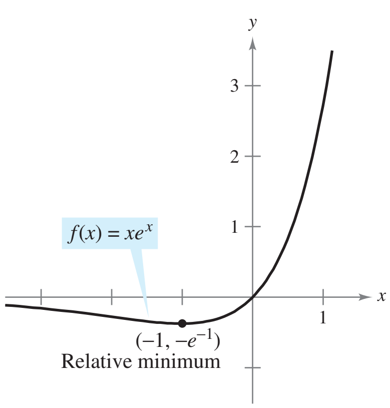
EXAMPLE 4 Locating Relative Extrema

Find the relative extrema of $f(x) = xe^x$.

Solution The derivative of f is given by

$$f'(x) = x(e^x) + e^x(1) \qquad \text{Product Rule}$$
$$= e^x(x + 1).$$

Because e^x is never 0, the derivative is 0 only when $x = -1$. Moreover, by the First Derivative Test, you can determine that this corresponds to a relative minimum, as shown in Figure 5.21. Because the derivative $f'(x) = e^x(x + 1)$ is defined for all x, there are no other critical points.



The derivative of f changes from negative to positive at $x = -1$.
Figure 5.21

EXAMPLE 5 The Standard Normal Probability Density Function

Show that the *standard normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.

Solution To locate possible points of inflection, find the x-values for which the second derivative is 0.

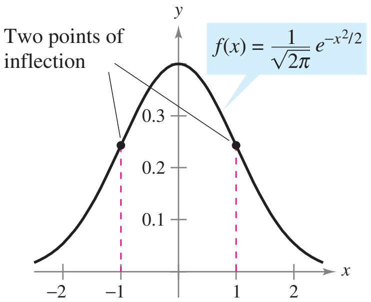
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$ Write original function.

$$f'(x) = \frac{1}{\sqrt{2\pi}}(-x)e^{-x^2/2}$$ First derivative

$$f''(x) = \frac{1}{\sqrt{2\pi}}\left[(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}\right]$$ Product Rule

$$= \frac{1}{\sqrt{2\pi}}\left(e^{-x^2/2}\right)(x^2 - 1)$$ Second derivative

So, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 3 to conclude that these values yield the two points of inflection shown in Figure 5.22.



The bell-shaped curve given by a standard normal probability density function
Figure 5.22

NOTE The general form of a normal probability density function (whose mean is 0) is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). This "bell-shaped curve" has points of inflection when $x = \pm\sigma$.

EXAMPLE 6 Shares Traded

The number y of shares traded (in millions) on the New York Stock Exchange from 1990 through 2002 can be modeled by

$$y = 36{,}663e^{0.1902t}$$

where t represents the year, with $t = 0$ corresponding to 1990. At what rate was the number of shares traded changing in 1998? *(Source: New York Stock Exchange, Inc.)*

Solution The derivative of the given model is

$$y' = (0.1902)(36{,}663)e^{0.1902t}$$
$$\approx 6973e^{0.1902t}.$$

By evaluating the derivative when $t = 8$, you can conclude that the rate of change in 1998 was about

31,933 million shares per year.

The graph of this model is shown in Figure 5.23.

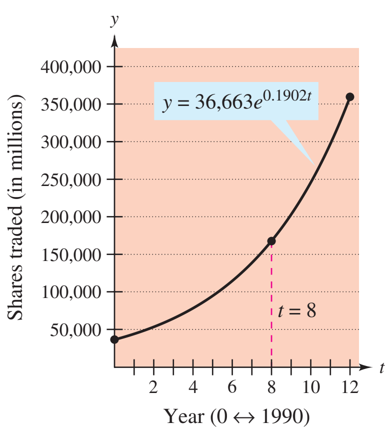


Figure 5.23

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\int e^x\,dx = e^x + C$ **2.** $\int e^u\,du = e^u + C$

EXAMPLE 7 Integrating Exponential Functions

Find $\int e^{3x+1}\,dx$.

Solution If you let $u = 3x + 1$, then $du = 3\,dx$.

$$\int e^{3x+1}dx = \frac{1}{3}\int e^{3x+1}(3)\,dx \qquad \text{Multiply and divide by 3.}$$

$$= \frac{1}{3}\int e^u\,du \qquad \text{Substitute: } u = 3x + 1.$$

$$= \frac{1}{3}e^u + C \qquad \text{Apply Exponential Rule.}$$

$$= \frac{e^{3x+1}}{3} + C \qquad \text{Back-substitute.}$$

NOTE In Example 7, the missing *constant* factor 3 was introduced to create $du = 3\,dx$. However, remember that you cannot introduce a missing *variable* factor in the integrand. For instance,

$$\int e^{-x^2}\,dx \neq \frac{1}{x}\int e^{-x^2}(x\,dx).$$

EXAMPLE 8 Integrating Exponential Functions

Find $\int 5xe^{-x^2}\,dx$.

Solution If you let $u = -x^2$, then $du = -2x\,dx$ or $x\,dx = -du/2$.

$$\int 5xe^{-x^2}dx = \int 5e^{-x^2}(x\,dx) \qquad \text{Regroup integrand.}$$

$$= \int 5e^u\left(-\frac{du}{2}\right) \qquad \text{Substitute: } u = -x^2.$$

$$= -\frac{5}{2}\int e^u\,du \qquad \text{Constant Multiple Rule}$$

$$= -\frac{5}{2}e^u + C \qquad \text{Apply Exponential Rule.}$$

$$= -\frac{5}{2}e^{-x^2} + C \qquad \text{Back-substitute.}$$

EXAMPLE 9 Integrating Exponential Functions

a. $$\int \frac{e^{1/x}}{x^2}\,dx = -\int \overbrace{e^{1/x}}^{e^u}\overbrace{\left(-\frac{1}{x^2}\right)dx}^{du} \qquad u = \frac{1}{x}$$

$$= -e^{1/x} + C$$

b. $$\int \sin x\, e^{\cos x}\,dx = -\int \overbrace{e^{\cos x}}^{e^u}\overbrace{(-\sin x\,dx)}^{du} \qquad u = \cos x$$

$$= -e^{\cos x} + C$$

EXAMPLE 10 Finding Areas Bounded by Exponential Functions

Evaluate each definite integral.

a. $\int_0^1 e^{-x}\,dx$ **b.** $\int_0^1 \frac{e^x}{1+e^x}\,dx$ **c.** $\int_{-1}^0 [e^x\cos(e^x)]\,dx$

Solution

a. $$\int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1 \qquad \text{See Figure 5.24(a).}$$

$$= -e^{-1} - (-1)$$

$$= 1 - \frac{1}{e}$$

$$\approx 0.632$$

b. $$\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)\Big]_0^1 \qquad \text{See Figure 5.24(b).}$$

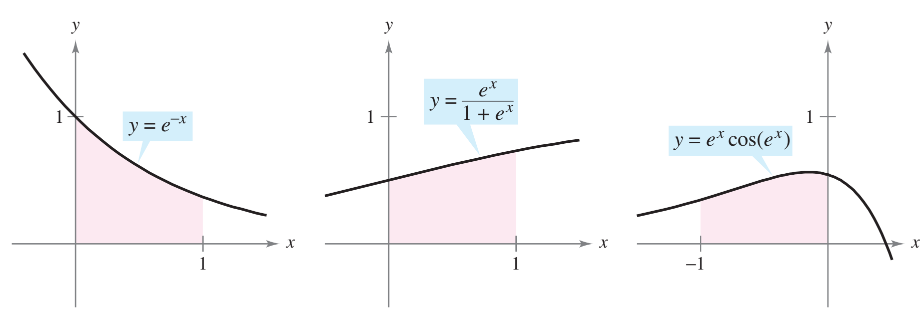
$$= \ln(1+e) - \ln 2$$

$$\approx 0.620$$

c. $$\int_{-1}^0 [e^x\cos(e^x)]\,dx = \sin(e^x)\Big]_{-1}^0 \qquad \text{See Figure 5.24(c).}$$

$$= \sin 1 - \sin(e^{-1})$$

$$\approx 0.482$$



(a) (b) (c)

Figure 5.24

Exercises for Section 5.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, solve for x accurate to three decimal places.

1. $e^{\ln x} = 4$
2. $e^{\ln 2x} = 12$
3. $e^x = 12$
4. $4e^x = 83$
5. $9 - 2e^x = 7$
6. $-6 + 3e^x = 8$
7. $50e^{-x} = 30$
8. $200e^{-4x} = 15$
9. $\ln x = 2$
10. $\ln x^2 = 10$
11. $\ln(x - 3) = 2$
12. $\ln 4x = 1$
13. $\ln\sqrt{x + 2} = 1$
14. $\ln(x - 2)^2 = 12$

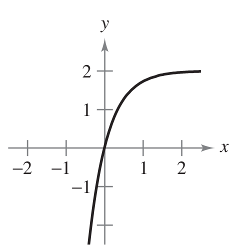
In Exercises 15–18, sketch the graph of the function.

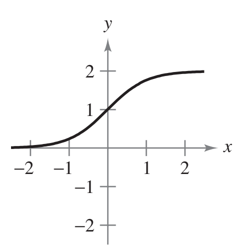
15. $y = e^{-x}$
16. $y = \frac{1}{2}e^x$
17. $y = e^{-x^2}$
18. $y = e^{-x/2}$

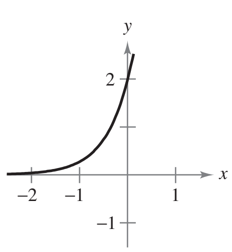
19. Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?
(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

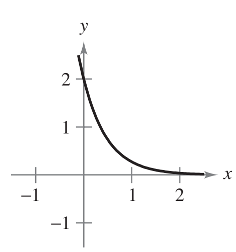
20. Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.
(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$
(b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

In Exercises 21–24, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]

(a)


(b)


(c)


(d)


21. $y = Ce^{ax}$
22. $y = Ce^{-ax}$
23. $y = C(1 - e^{-ax})$
24. $y = \dfrac{C}{1 + e^{-ax}}$

In Exercises 25–28, illustrate that the functions are inverses of each other by graphing both functions on the same set of coordinate axes.

25. $f(x) = e^{2x}$
$g(x) = \ln\sqrt{x}$
26. $f(x) = e^{x/3}$
$g(x) = \ln x^3$
27. $f(x) = e^x - 1$
$g(x) = \ln(x + 1)$
28. $f(x) = e^{x-1}$
$g(x) = 1 + \ln x$

29. ***Graphical Analysis*** Use a graphing utility to graph

$$f(x) = \left(1 + \frac{0.5}{x}\right)^x \quad \text{and} \quad g(x) = e^{0.5}$$

in the same viewing window. What is the relationship between f and g as $x \to \infty$?

30. ***Conjecture*** Use the result of Exercise 29 to make a conjecture about the value of

$$\left(1 + \frac{r}{x}\right)^x$$

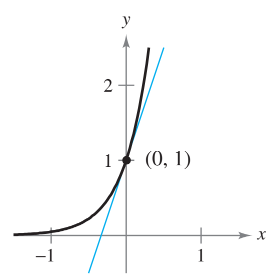
as $x \to \infty$.

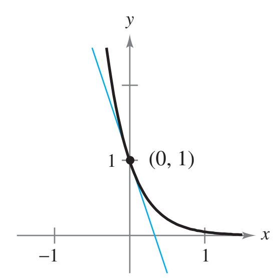
In Exercises 31 and 32, compare the given number with the number e. Is the number less than or greater than e?

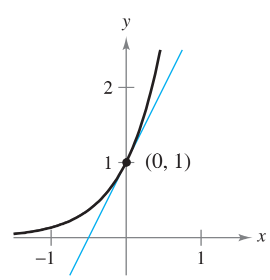
31. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$ (See Exercise 30.)

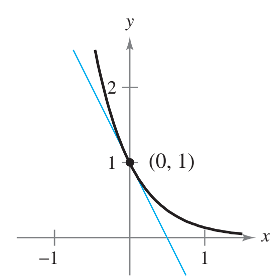
32. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040}$

In Exercises 33 and 34, find an equation of the tangent line to the graph of the function at the point (0, 1).

33. (a) $y = e^{3x}$


(b) $y = e^{-3x}$


34. (a) $y = e^{2x}$


(b) $y = e^{-2x}$


In Exercises 35–48, find the derivative.

35. $f(x) = e^{2x}$

36. $y = e^{-x^2}$

37. $y = e^{\sqrt{x}}$

38. $y = x^2e^{-x}$

39. $g(t) = (e^{-t} + e^t)^3$

40. $g(t) = e^{-3/t^2}$

41. $y = \ln(1 + e^{2x})$

42. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$

43. $y = \dfrac{2}{e^x + e^{-x}}$

44. $y = \dfrac{e^x - e^{-x}}{2}$

45. $y = e^x(\sin x + \cos x)$

46. $y = \ln e^x$

47. $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t\,dt$

48. $F(x) = \displaystyle\int_0^{e^{2x}} \ln(t + 1)\,dt$

In Exercises 49–56, find an equation of the tangent line to the graph of the function at the given point.

49. $f(x) = e^{1-x}$, $(1, 1)$

50. $y = e^{-2x+x^2}$, $(2, 1)$

51. $y = \ln(e^{x^2})$, $(-2, 4)$

52. $y = \ln\dfrac{e^x + e^{-x}}{2}$, $(0, 0)$

53. $y = x^2e^x - 2xe^x + 2e^x$, $(1, e)$

54. $y = xe^x - e^x$, $(1, 0)$

55. $f(x) = e^{-x}\ln x$, $(1, 0)$

56. $f(x) = e^3 \ln x$, $(1, 0)$

In Exercises 57 and 58, use implicit differentiation to find dy/dx.

57. $xe^y - 10x + 3y = 0$

58. $e^{xy} + x^2 - y^2 = 10$

In Exercises 59 and 60, find an equation of the tangent line to the graph of the function at the given point.

59. $xe^y + ye^x = 1$, $(0, 1)$

60. $1 + \ln xy = e^{x-y}$, $(1, 1)$

In Exercises 61 and 62, find the second derivative of the function.

61. $f(x) = (3 + 2x)e^{-3x}$

62. $g(x) = \sqrt{x} + e^x \ln x$

In Exercises 63 and 64, show that the function $y = f(x)$ is a solution of the differential equation.

63. $y = e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$

$y'' - 2y' + 3y = 0$

64. $y = e^x(3\cos 2x - 4\sin 2x)$

$y'' - 2y' + 5y = 0$

In Exercises 65–72, find the extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

65. $f(x) = \dfrac{e^x + e^{-x}}{2}$

66. $f(x) = \dfrac{e^x - e^{-x}}{2}$

67. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

68. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

69. $f(x) = x^2e^{-x}$

70. $f(x) = xe^{-x}$

71. $g(t) = 1 + (2 + t)e^{-t}$

72. $f(x) = -2 + e^{3x}(4 - 2x)$

73. ***Area*** Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

74. ***Area*** Perform the following steps to find the maximum area of the rectangle shown in the figure.

(a) Solve for c in the equation $f(c) = f(c + x)$.

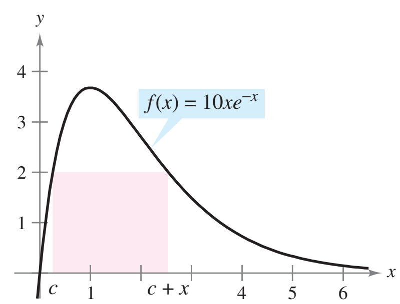
(b) Use the result in part (a) to write the area A as a function of x. [*Hint:* $A = xf(c)$]

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the maximum area.

(d) Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate

$$\lim_{x\to 0^+} c \quad \text{and} \quad \lim_{x\to\infty} c.$$

Use this result to describe the changes in dimensions and position of the rectangle for $0 < x < \infty$.



75. Verify that the function

$$y = \frac{L}{1 + ae^{-x/b}}, \quad a > 0, \quad b > 0, \quad L > 0$$

increases at a maximum rate when $y = L/2$.

76. ***Writing*** Consider the function $f(x) = \dfrac{2}{1 + e^{1/x}}$.

(a) Use a graphing utility to graph f.

(b) Write a short paragraph explaining why the graph has a horizontal asymptote at $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

77. Find a point on the graph of the function $f(x) = e^{2x}$ such that the tangent line to the graph at that point passes through the origin. Use a graphing utility to graph f and the tangent line in the same viewing window.

78. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve passes through the origin. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

79. ***Depreciation*** The value V of an item t years after it is purchased is $V = 15{,}000e^{-0.6286t}$, $0 \le t \le 10$.

(a) Use a graphing utility to graph the function.

(b) Find the rate of change of V with respect to t when $t = 1$ and $t = 5$.

(c) Use a graphing utility to graph the tangent line to the function when $t = 1$ and $t = 5$.

80. ***Harmonic Motion*** The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is

$y = 1.56e^{-0.22t} \cos 4.9t$

where y is the displacement in feet and t is the time in seconds. Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

81. ***Modeling Data*** A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517

(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form

$\ln P = ah + b.$

Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rate of change of the pressure when $h = 5$ and $h = 18$.

82. ***Modeling Data*** The table lists the approximate value V of a mid-sized sedan for the years 1997 through 2003. The variable t represents the time in years, with $t = 7$ corresponding to 1997.

t	7	8	9	10
V	\$17,040	\$14,590	\$12,845	\$10,995

t	11	12	13
V	\$9220	\$8095	\$6835

(a) Use a computer algebra system to find linear and quadratic models for the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use a computer algebra system to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Find the rate of decrease in the value of the sedan when $t = 8$ and $t = 12$ using the exponential model.

Linear and Quadratic Approximations **In Exercises 83 and 84, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(0) + f'(0)(x - 0)$$

and

$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 0$.

83. $f(x) = e^{x/2}$

84. $f(x) = e^{-x^2/2}$

In Exercises 85–98, find the indefinite integral.

85. $\int e^{5x}(5)\, dx$

86. $\int e^{-x^4}(-4x^3)\, dx$

87. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

88. $\int \frac{e^{1/x^2}}{x^3}\, dx$

89. $\int \frac{e^{-x}}{1 + e^{-x}}\, dx$

90. $\int \frac{e^{2x}}{1 + e^{2x}}\, dx$

91. $\int e^x\sqrt{1 - e^x}\, dx$

92. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

93. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$

94. $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\, dx$

95. $\int \frac{5 - e^x}{e^{2x}}\, dx$

96. $\int \frac{e^{2x} + 2e^x + 1}{e^x}\, dx$

97. $\int e^{-x} \tan(e^{-x})\, dx$

98. $\int \ln(e^{2x-1})\, dx$

In Exercises 99–106, evaluate the definite integral. Use a graphing utility to verify your result.

99. $\int_0^1 e^{-2x}\, dx$

100. $\int_3^4 e^{3-x}\, dx$

101. $\int_0^1 xe^{-x^2}\, dx$

102. $\int_{-2}^0 x^2 e^{x^3/2}\, dx$

103. $\int_1^3 \frac{e^{3/x}}{x^2}\, dx$

104. $\int_0^{\sqrt{2}} xe^{-(x^2/2)}\, dx$

105. $\int_0^{\pi/2} e^{\sin \pi x} \cos \pi x\, dx$

106. $\int_{\pi/3}^{\pi/2} e^{\sec 2x} \sec 2x \tan 2x\, dx$

Differential Equations **In Exercises 107 and 108, solve the differential equation.**

107. $\frac{dy}{dx} = xe^{ax^2}$

108. $\frac{dy}{dx} = (e^x - e^{-x})^2$

Differential Equations **In Exercises 109 and 110, find the particular solution that satisfies the initial conditions.**

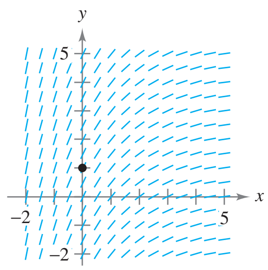
109. $f''(x) = \frac{1}{2}(e^x + e^{-x})$, $f(0) = 1, f'(0) = 0$

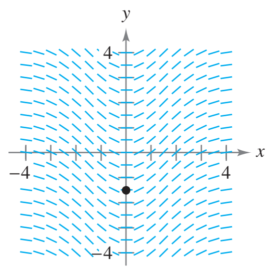
110. $f''(x) = \sin x + e^{2x}$, $f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

Slope Fields **In Exercises 111 and 112, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

111. $\dfrac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

112. $\dfrac{dy}{dx} = xe^{-0.2x^2}$, $\left(0, -\dfrac{3}{2}\right)$





Area **In Exercises 113–116, find the area of the region bounded by the graphs of the equations. Use a graphing utility to graph the region and verify your result.**

113. $y = e^x, y = 0, x = 0, x = 5$

114. $y = e^{-x}, y = 0, x = a, x = b$

115. $y = xe^{-x^2/4}, y = 0, x = 0, x = \sqrt{6}$

116. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

Numerical Integration **In Exercises 117 and 118, approximate the integral using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with $n = 12$. Use a graphing utility to verify your results.**

117. $\displaystyle\int_0^4 \sqrt{x}\, e^x\, dx$

118. $\displaystyle\int_0^2 2xe^{-x}\, dx$

119. ***Probability*** A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is

$$0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2}\, dt.$$

Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

120. ***Probability*** The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation

$$\int_0^x 0.3e^{-0.3t}\, dt = \frac{1}{2}.$$

Solve the equation.

121. Given $e^x \geq 1$ for $x \geq 0$, it follows that

$$\int_0^x e^t\, dt \geq \int_0^x 1\, dt.$$

Perform this integration to derive the inequality $e^x \geq 1 + x$ for $x \geq 0$.

122. ***Modeling Data*** A valve on a storage tank is opened for 4 hours to release a chemical in a manufacturing process. The flow rate R (in liters per hour) at time t (in hours) is given in the table.

t	0	1	2	3	4
R	425	240	118	71	36

(a) Use the regression capabilities of a graphing utility to find a linear model for the points $(t, \ln R)$. Write the resulting equation of the form $\ln R = at + b$ in exponential form.

(b) Use a graphing utility to plot the data and graph the exponential model.

(c) Use the definite integral to approximate the number of liters of chemical released during the 4 hours.

Writing About Concepts

123. In your own words, state the properties of the natural exponential function.

124. Describe the relationship between the graphs of $f(x) = \ln x$ and $g(x) = e^x$.

125. Is there a function f such that $f(x) = f'(x)$? If so, identify it.

126. Without integrating, state the integration formula you can use to integrate each of the following.

(a) $\displaystyle\int \frac{e^x}{e^x + 1}\, dx$ (b) $\displaystyle\int xe^{x^2}\, dx$

127. Find, to three decimal places, the value of x such that $e^{-x} = x$. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

128. Find the value of a such that the area bounded by $y = e^{-x}$, the x-axis, $x = -a$, and $x = a$ is $\frac{8}{3}$.

129. Prove that $\dfrac{e^a}{e^b} = e^{a-b}$.

130. Let $f(x) = \dfrac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreasing on (e, ∞).

(b) Show that if $e \leq A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.