

Section 7.1

Area of a Region Between Two Curves

- Find the area of a region between two curves using integration.
- Find the area of a region between intersecting curves using integration.
- Describe integration as an accumulation process.

Area of a Region Between Two Curves

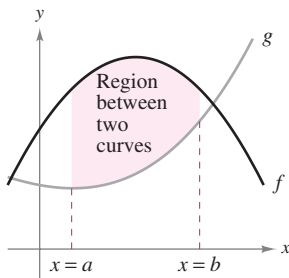


Figure 7.1

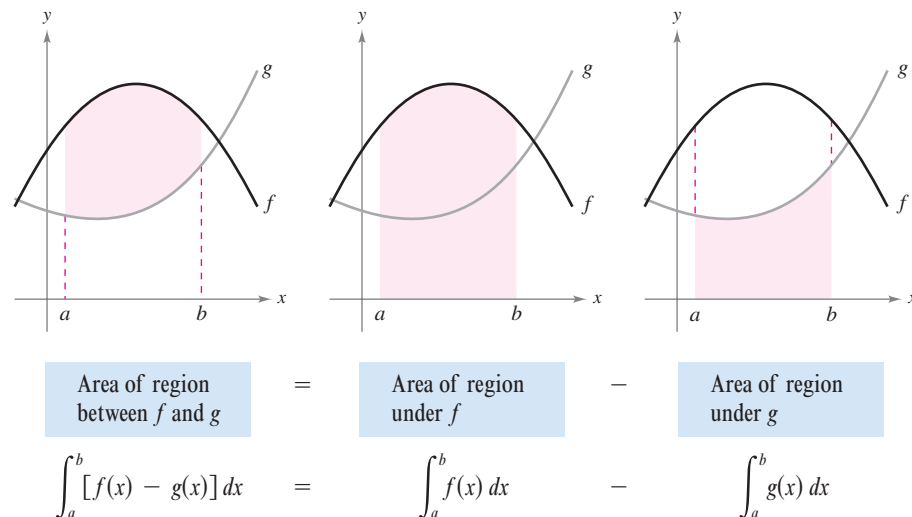


Figure 7.2

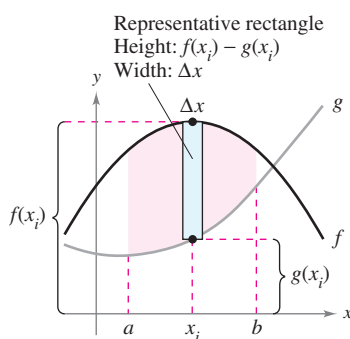


Figure 7.3

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx . Then, as shown in Figure 7.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the i th interval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)] \Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \rightarrow 0$ ($n \rightarrow \infty$), you obtain

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n [f(x_i) - g(x_i)] \Delta x.$$

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n [f(x_i) - g(x_i)] \Delta x \\ &= \int_a^b [f(x) - g(x)] dx. \end{aligned}$$

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx.$$

In Figure 7.1, the graphs of f and g are shown above the x -axis. This, however, is not necessary. The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ for all x in the interval $[a, b]$. This result is summarized graphically in Figure 7.4.

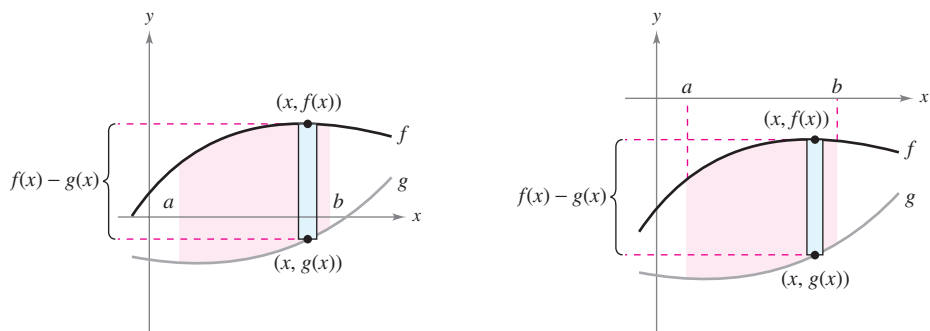


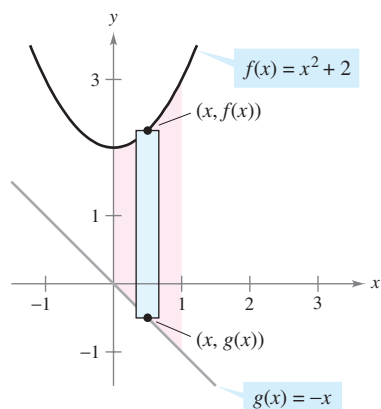
Figure 7.4

NOTE The height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x -axis, as shown in Figure 7.4.

Representative rectangles are used throughout this chapter in various applications of integration. A vertical rectangle (of width Δx) implies integration with respect to x , whereas a horizontal rectangle (of width Δy) implies integration with respect to y .

EXAMPLE 1 Finding the Area of a Region Between Two Curves

Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.



Region bounded by the graph of f , the graph of g , $x = 0$, and $x = 1$

Figure 7.5

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \leq f(x)$ for all x in $[0, 1]$, as shown in Figure 7.5. So, the area of the representative rectangle is

$$\begin{aligned} \Delta A &= [f(x) - g(x)] \Delta x \\ &= [(x^2 + 2) - (-x)] \Delta x \end{aligned}$$

and the area of the region is

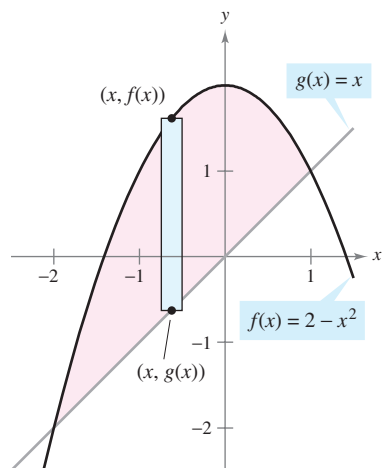
$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] dx = \int_0^1 [(x^2 + 2) - (-x)] dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1 \\ &= \frac{1}{3} + \frac{1}{2} + 2 \\ &= \frac{17}{6}. \end{aligned}$$

Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly. A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of a and b must be calculated.

EXAMPLE 2 A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.



Region bounded by the graph of f and the graph of g
Figure 7.6

Solution In Figure 7.6, notice that the graphs of f and g have two points of intersection. To find the x -coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x .

$$\begin{aligned} 2 - x^2 &= x && \text{Set } f(x) \text{ equal to } g(x). \\ -x^2 - x + 2 &= 0 && \text{Write in general form.} \\ -(x + 2)(x - 1) &= 0 && \text{Factor.} \\ x &= -2 \text{ or } 1 && \text{Solve for } x. \end{aligned}$$

So, $a = -2$ and $b = 1$. Because $g(x) \leq f(x)$ for all x in the interval $[-2, 1]$, the representative rectangle has an area of

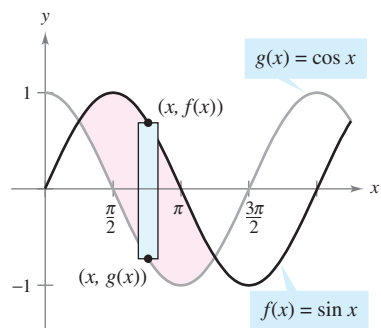
$$\begin{aligned} \Delta A &= [f(x) - g(x)] \Delta x \\ &= [(2 - x^2) - x] \Delta x \end{aligned}$$

and the area of the region is

$$\begin{aligned} A &= \int_{-2}^1 [(2 - x^2) - x] dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\ &= \frac{9}{2}. \end{aligned}$$

EXAMPLE 3 A Region Lying Between Two Intersecting Graphs

The sine and cosine curves intersect infinitely many times, bounding regions of equal areas, as shown in Figure 7.7. Find the area of one of these regions.



One of the regions bounded by the graphs of the sine and cosine functions
Figure 7.7

Solution

$$\begin{aligned} \sin x &= \cos x && \text{Set } f(x) \text{ equal to } g(x). \\ \frac{\sin x}{\cos x} &= 1 && \text{Divide each side by } \cos x. \\ \tan x &= 1 && \text{Trigonometric identity} \\ x &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \quad 0 \leq x \leq 2\pi && \text{Solve for } x. \end{aligned}$$

So, $a = \pi/4$ and $b = 5\pi/4$. Because $\sin x \geq \cos x$ for all x in the interval $[\pi/4, 5\pi/4]$, the area of the region is

$$\begin{aligned} A &= \int_{\pi/4}^{5\pi/4} [\sin x - \cos x] dx = \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4} \\ &= 2\sqrt{2}. \end{aligned}$$

If two curves intersect at more than two points, then to find the area of the region between the curves, you must find all points of intersection and check to see which curve is above the other in each interval determined by these points.



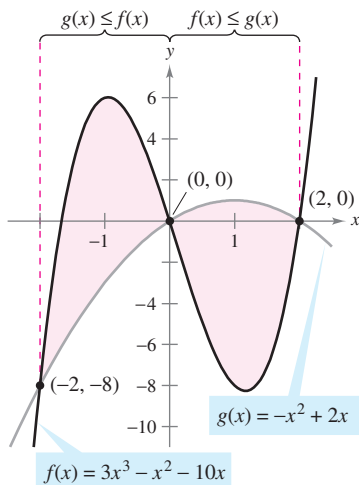
EXAMPLE 4 Curves That Intersect at More Than Two Points

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

Solution Begin by setting $f(x)$ and $g(x)$ equal to each other and solving for x . This yields the x -values at each point of intersection of the two graphs.

$$\begin{aligned} 3x^3 - x^2 - 10x &= -x^2 + 2x && \text{Set } f(x) \text{ equal to } g(x). \\ 3x^3 - 12x &= 0 && \text{Write in general form.} \\ 3x(x-2)(x+2) &= 0 && \text{Factor.} \\ x &= -2, 0, 2 && \text{Solve for } x. \end{aligned}$$

So, the two graphs intersect when $x = -2, 0,$ and 2 . In Figure 7.8, notice that $g(x) \leq f(x)$ on the interval $[-2, 0]$. However, the two graphs switch at the origin, and $f(x) \leq g(x)$ on the interval $[0, 2]$. So, you need two integrals—one for the interval $[-2, 0]$ and one for the interval $[0, 2]$.



On $[-2, 0]$, $g(x) \leq f(x)$, and on $[0, 2]$, $f(x) \leq g(x)$.

Figure 7.8

$$\begin{aligned} A &= \int_{-2}^0 [f(x) - g(x)] dx + \int_0^2 [g(x) - f(x)] dx \\ &= \int_{-2}^0 (3x^3 - 12x) dx + \int_0^2 (-3x^3 + 12x) dx \\ &= \left[\frac{3x^4}{4} - 6x^2 \right]_{-2}^0 + \left[-\frac{3x^4}{4} + 6x^2 \right]_0^2 \\ &= -(12 - 24) + (-12 + 24) = 24 \end{aligned}$$

NOTE In Example 4, notice that you obtain an incorrect result if you integrate from -2 to 2 . Such integration produces

$$\int_{-2}^2 [f(x) - g(x)] dx = \int_{-2}^2 (3x^3 - 12x) dx = 0.$$

If the graph of a function of y is a boundary of a region, it is often convenient to use representative rectangles that are *horizontal* and find the area by integrating with respect to y . In general, to determine the area between two curves, you can use

$$A = \int_{x_1}^{x_2} \underbrace{[(\text{top curve}) - (\text{bottom curve})]}_{\text{in variable } x} dx \quad \text{Vertical rectangles}$$

$$A = \int_{y_1}^{y_2} \underbrace{[(\text{right curve}) - (\text{left curve})]}_{\text{in variable } y} dy \quad \text{Horizontal rectangles}$$

where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

indicates that in the HM mathSpace® CD-ROM and the online Eduspace® system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems Maple, Mathcad, Mathematica, and Derive.

EXAMPLE 5 Horizontal Representative Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Solution Consider

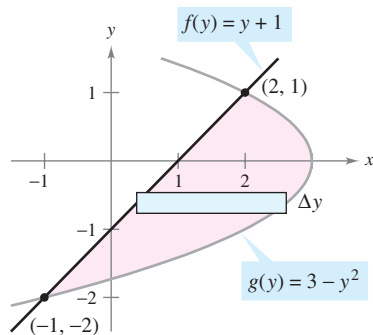
$$g(y) = 3 - y^2 \quad \text{and} \quad f(y) = y + 1.$$

These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure 7.9. Because $f(y) \leq g(y)$ on this interval, you have

$$\Delta A = [g(y) - f(y)] \Delta y = [(3 - y^2) - (y + 1)] \Delta y.$$

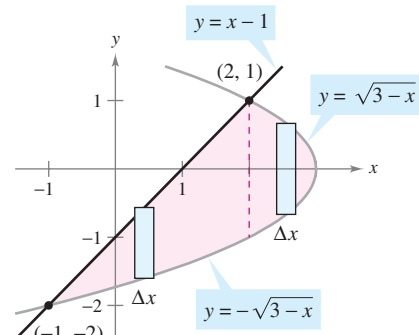
So, the area is

$$\begin{aligned} A &= \int_{-2}^1 [(3 - y^2) - (y + 1)] dy \\ &= \int_{-2}^1 (-y^2 - y + 2) dy \\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y \right]_{-2}^1 \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\ &= \frac{9}{2}. \end{aligned}$$



Horizontal rectangles (integration with respect to y)

Figure 7.9



Vertical rectangles (integration with respect to x)

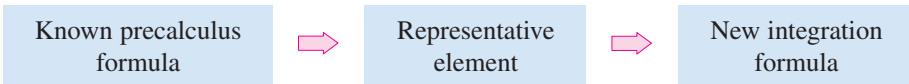
Figure 7.10

In Example 5, notice that by integrating with respect to y you need only one integral. If you had integrated with respect to x , you would have needed two integrals because the upper boundary would have changed at $x = 2$, as shown in Figure 7.10.

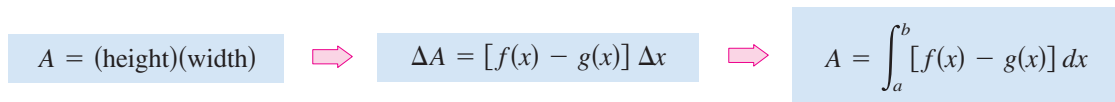
$$\begin{aligned} A &= \int_{-1}^2 [(x - 1) + \sqrt{3 - x}] dx + \int_2^3 (\sqrt{3 - x} + \sqrt{3 - x}) dx \\ &= \int_{-1}^2 [x - 1 + (3 - x)^{1/2}] dx + 2 \int_2^3 (3 - x)^{1/2} dx \\ &= \left[\frac{x^2}{2} - x - \frac{(3 - x)^{3/2}}{3/2} \right]_{-1}^2 - 2 \left[\frac{(3 - x)^{3/2}}{3/2} \right]_2^3 \\ &= \left(2 - 2 - \frac{2}{3} \right) - \left(\frac{1}{2} + 1 - \frac{16}{3} \right) - 2(0) + 2 \left(\frac{2}{3} \right) \\ &= \frac{9}{2} \end{aligned}$$

Integration as an Accumulation Process

In this section, the integration formula for the area between two curves was developed by using a rectangle as the *representative element*. For each new application in the remaining sections of this chapter, an appropriate representative element will be constructed using precalculus formulas you already know. Each integration formula will then be obtained by summing or accumulating these representative elements.



For example, in this section the area formula was developed as follows.



EXAMPLE 6 Describing Integration as an Accumulation Process

Find the area of the region bounded by the graph of $y = 4 - x^2$ and the x -axis. Describe the integration as an accumulation process.

Solution The area of the region is given by

$$A = \int_{-2}^2 (4 - x^2) dx.$$

You can think of the integration as an accumulation of the areas of the rectangles formed as the representative rectangle slides from $x = -2$ to $x = 2$, as shown in Figure 7.11.

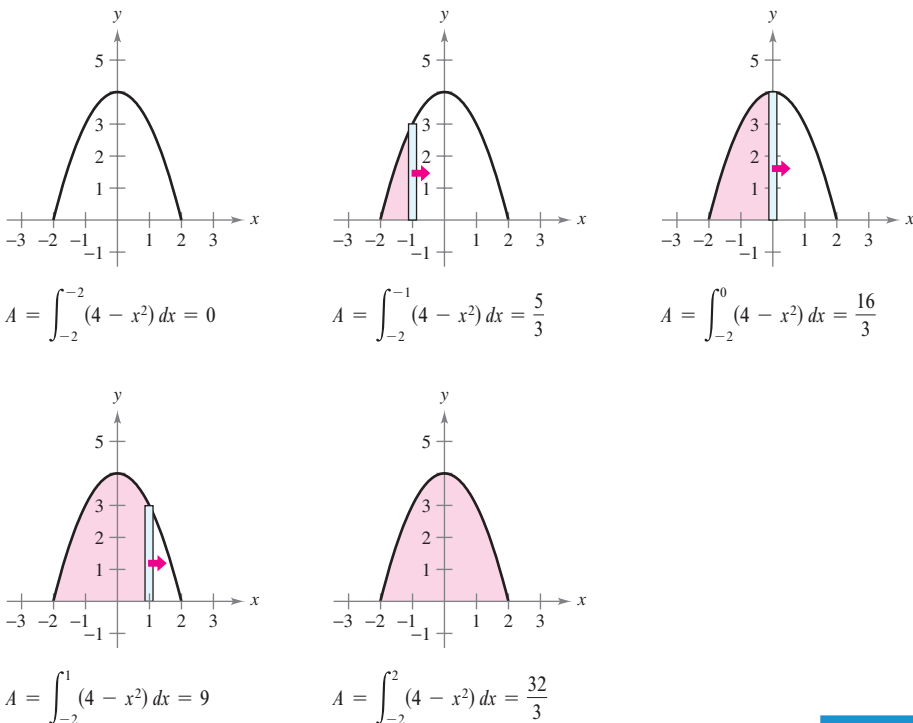


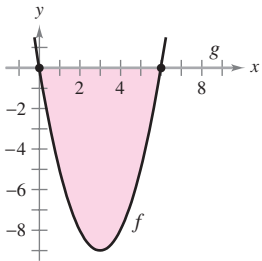
Figure 7.11

Exercises for Section 7.1

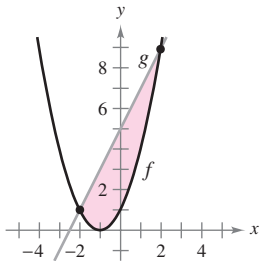
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, set up the definite integral that gives the area of the region.

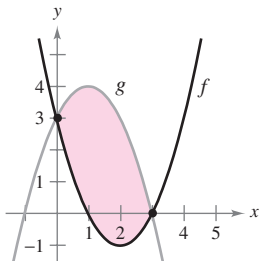
1. $f(x) = x^2 - 6x$
 $g(x) = 0$



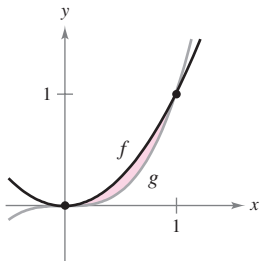
2. $f(x) = x^2 + 2x + 1$
 $g(x) = 2x + 5$



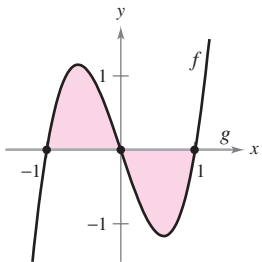
3. $f(x) = x^2 - 4x + 3$
 $g(x) = -x^2 + 2x + 3$



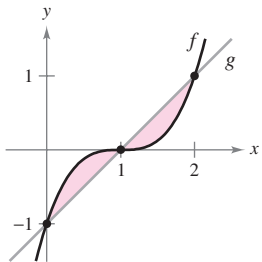
4. $f(x) = x^2$
 $g(x) = x^3$



5. $f(x) = 3(x^3 - x)$
 $g(x) = 0$



6. $f(x) = (x - 1)^3$
 $g(x) = x - 1$



In Exercises 7–12, the integrand of the definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.

7. $\int_0^4 \left[(x + 1) - \frac{x}{2} \right] dx$

8. $\int_{-1}^1 [(1 - x^2) - (x^2 - 1)] dx$

9. $\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6} \right] dx$

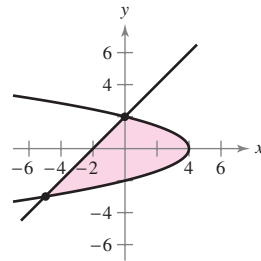
11. $\int_{-\pi/3}^{\pi/3} (2 - \sec x) dx$

10. $\int_2^3 \left[\left(\frac{x^3}{3} - x \right) - \frac{x}{3} \right] dx$

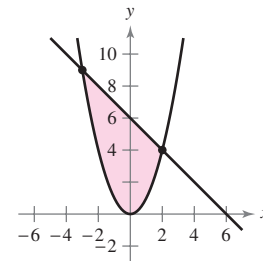
12. $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x) dx$

In Exercises 13 and 14, find the area of the region by integrating (a) with respect to x and (b) with respect to y .

13. $x = 4 - y^2$
 $x = y - 2$



14. $y = x^2$
 $y = 6 - x$



Think About It In Exercises 15 and 16, determine which value best approximates the area of the region bounded by the graphs of f and g . (Make your selection on the basis of a sketch of the region and not by performing any calculations.)

15. $f(x) = x + 1$, $g(x) = (x - 1)^2$

- (a) -2 (b) 2 (c) 10 (d) 4 (e) 8

16. $f(x) = 2 - \frac{1}{2}x$, $g(x) = 2 - \sqrt{x}$

- (a) 1 (b) 6 (c) -3 (d) 3 (e) 4

In Exercises 17–32, sketch the region bounded by the graphs of the algebraic functions and find the area of the region.

17. $y = \frac{1}{2}x^3 + 2$, $y = x + 1$, $x = 0$, $x = 2$

18. $y = -\frac{3}{8}x(x - 8)$, $y = 10 - \frac{1}{2}x$, $x = 2$, $x = 8$

19. $f(x) = x^2 - 4x$, $g(x) = 0$

20. $f(x) = -x^2 + 4x + 1$, $g(x) = x + 1$

21. $f(x) = x^2 + 2x + 1$, $g(x) = 3x + 3$

22. $f(x) = -x^2 + 4x + 2$, $g(x) = x + 2$

23. $y = x$, $y = 2 - x$, $y = 0$

24. $y = \frac{1}{x^2}$, $y = 0$, $x = 1$, $x = 5$

25. $f(x) = \sqrt{3x} + 1$, $g(x) = x + 1$

26. $f(x) = \sqrt[3]{x - 1}$, $g(x) = x - 1$

27. $f(y) = y^2$, $g(y) = y + 2$


28. $f(y) = y(2 - y)$, $g(y) = -y$

29. $f(y) = y^2 + 1$, $g(y) = 0$, $y = -1$, $y = 2$

30. $f(y) = \frac{y}{\sqrt{16 - y^2}}$, $g(y) = 0$, $y = 3$

31. $f(x) = \frac{10}{x}$, $x = 0$, $y = 2$, $y = 10$

32. $g(x) = \frac{4}{2 - x}$, $y = 4$, $x = 0$

 In Exercises 33–42, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

33. $f(x) = x(x^2 - 3x + 3)$, $g(x) = x^2$

34. $f(x) = x^3 - 2x + 1$, $g(x) = -2x$, $x = 1$

35. $y = x^2 - 4x + 3$, $y = 3 + 4x - x^2$

36. $y = x^4 - 2x^2$, $y = 2x^2$

37. $f(x) = x^4 - 4x^2$, $g(x) = x^2 - 4$

38. $f(x) = x^4 - 4x^2$, $g(x) = x^3 - 4x$

39. $f(x) = 1/(1 + x^2)$, $g(x) = \frac{1}{2}x^2$

40. $f(x) = 6x/(x^2 + 1)$, $y = 0$, $0 \leq x \leq 3$

41. $y = \sqrt{1 + x^3}$, $y = \frac{1}{2}x + 2$, $x = 0$

42. $y = x\sqrt{\frac{4-x}{4+x}}$, $y = 0$, $x = 4$

In Exercises 43–48, sketch the region bounded by the graphs of the functions, and find the area of the region.

43. $f(x) = 2 \sin x$, $g(x) = \tan x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$


44. $f(x) = \sin x$, $g(x) = \cos 2x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{6}$

45. $f(x) = \cos x$, $g(x) = 2 - \cos x$, $0 \leq x \leq 2\pi$

46. $f(x) = \sec \frac{\pi x}{4} \tan \frac{\pi x}{4}$, $g(x) = (\sqrt{2} - 4)x + 4$, $x = 0$

47. $f(x) = xe^{-x^2}$, $y = 0$, $0 \leq x \leq 1$

48. $f(x) = 3^x$, $g(x) = 2x + 1$


 In Exercises 49–52, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

49. $f(x) = 2 \sin x + \sin 2x$, $y = 0$, $0 \leq x \leq \pi$

50. $f(x) = 2 \sin x + \cos 2x$, $y = 0$, $0 < x \leq \pi$

51. $f(x) = \frac{1}{x^2}e^{1/x}$, $y = 0$, $1 \leq x \leq 3$

52. $g(x) = \frac{4 \ln x}{x}$, $y = 0$, $x = 5$

 In Exercises 53–56, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) explain why the area of the region is difficult to find by hand, and (c) use the integration capabilities of the graphing utility to approximate the area to four decimal places.

53. $y = \sqrt{\frac{x^3}{4-x}}$, $y = 0$, $x = 3$

54. $y = \sqrt{x}e^x$, $y = 0$, $x = 0$, $x = 1$

55. $y = x^2$, $y = 4 \cos x$

56. $y = x^2$, $y = \sqrt{3+x}$

In Exercises 57–60, find the accumulation function F . Then evaluate F at each value of the independent variable and graphically show the area given by each value of F .

57. $F(x) = \int_0^x (\frac{1}{2}t + 1) dt$ (a) $F(0)$ (b) $F(2)$ (c) $F(6)$

58. $F(x) = \int_0^x (\frac{1}{2}t^2 + 2) dt$ (a) $F(0)$ (b) $F(4)$ (c) $F(6)$

59. $F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2} d\theta$ (a) $F(-1)$ (b) $F(0)$ (c) $F(\frac{1}{2})$

60. $F(y) = \int_{-1}^y 4e^{x/2} dx$ (a) $F(-1)$ (b) $F(0)$ (c) $F(4)$

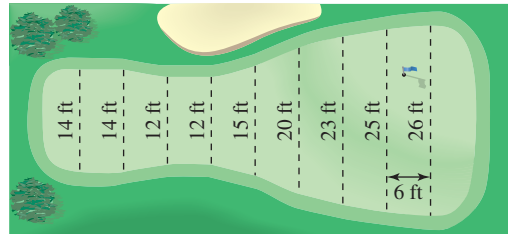
In Exercises 61–64, use integration to find the area of the figure having the given vertices.

61. $(2, -3)$, $(4, 6)$, $(6, 1)$ 62. $(0, 0)$, $(a, 0)$, (b, c)

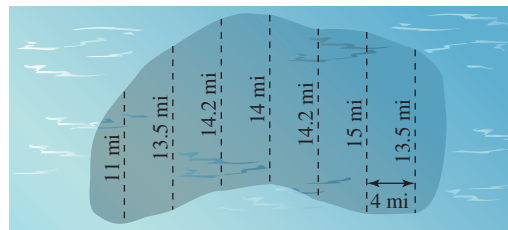
63. $(0, 2)$, $(4, 2)$, $(0, -2)$, $(-4, -2)$

64. $(0, 0)$, $(1, 2)$, $(3, -2)$, $(1, -3)$

65. **Numerical Integration** Estimate the surface area of the golf green using (a) the Trapezoidal Rule and (b) Simpson's Rule.



66. **Numerical Integration** Estimate the surface area of the oil spill using (a) the Trapezoidal Rule and (b) Simpson's Rule.



In Exercises 67–70, set up and evaluate the definite integral that gives the area of the region bounded by the graph of the function and the tangent line to the graph at the given point.

67. $f(x) = x^3$, $(1, 1)$ 68. $y = x^3 - 2x$, $(-1, 1)$

69. $f(x) = \frac{1}{x^2 + 1}$, $(1, \frac{1}{2})$ 70. $y = \frac{2}{1 + 4x^2}$, $(\frac{1}{2}, 1)$

Writing About Concepts

71. The graphs of $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ intersect at three points. However, the area between the curves can be found by a single integral. Explain why this is so, and write an integral for this area.

Writing About Concepts (continued)

72. The area of the region bounded by the graphs of $y = x^3$ and $y = x$ cannot be found by the single integral $\int_{-1}^1 (x^3 - x) dx$. Explain why this is so. Use symmetry to write a single integral that does represent the area.

73. A college graduate has two job offers. The starting salary for each is \$32,000, and after 8 years of service each will pay \$54,000. The salary increase for each offer is shown in the figure. From a strictly monetary viewpoint, which is the better offer? Explain.

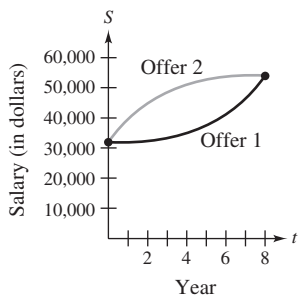


Figure for 73

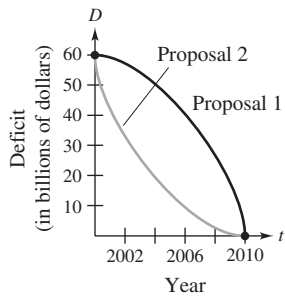


Figure for 74

74. A state legislature is debating two proposals for eliminating the annual budget deficits by the year 2010. The rate of decrease of the deficits for each proposal is shown in the figure. From the viewpoint of minimizing the cumulative state deficit, which is the better proposal? Explain.

In Exercises 75 and 76, find b such that the line $y = b$ divides the region bounded by the graphs of the two equations into two regions of equal area.

75. $y = 9 - x^2$, $y = 0$ 76. $y = 9 - |x|$, $y = 0$

In Exercises 77 and 78, find a such that the line $x = a$ divides the region bounded by the graphs of the equations into two regions of equal area.

77. $y = x$, $y = 4$, $x = 0$ 78. $y^2 = 4 - x$, $x = 0$


In Exercises 79 and 80, evaluate the limit and sketch the graph of the region whose area is represented by the limit.

79. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (x_i - x_i^2) \Delta x$, where $x_i = i/n$ and $\Delta x = 1/n$

80. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (4 - x_i^2) \Delta x$, where $x_i = -2 + (4i/n)$ and $\Delta x = 4/n$

Revenue In Exercises 81 and 82, two models R_1 and R_2 are given for revenue (in billions of dollars per year) for a large corporation. The model R_1 gives projected annual revenues from 2000 to 2005, with $t = 0$ corresponding to 2000, and R_2 gives projected revenues if there is a decrease in the rate of growth of corporate sales over the period. Approximate the total reduction in revenue if corporate sales are actually closer to the model R_2 .


81. $R_1 = 7.21 + 0.58t$ 82. $R_1 = 7.21 + 0.26t + 0.02t^2$
 $R_2 = 7.21 + 0.45t$ $R_2 = 7.21 + 0.1t + 0.01t^2$

 83. **Modeling Data** The table shows the total receipts R and total expenditures E for the Old-Age and Survivors Insurance Trust Fund (Social Security Trust Fund) in billions of dollars. The time t is given in years, with $t = 1$ corresponding to 1991. (Source: Social Security Administration)

t	1	2	3	4	5	6
R	299.3	311.2	323.3	328.3	342.8	363.7
E	245.6	259.9	273.1	284.1	297.8	308.2

t	7	8	9	10	11
R	397.2	424.8	457.0	490.5	518.1
E	322.1	332.3	339.9	358.3	377.5

- Use a graphing utility to fit an exponential model to the data for receipts. Plot the data and graph the model.
- Use a graphing utility to fit an exponential model to the data for expenditures. Plot the data and graph the model.
- If the models are assumed to be true for the years 2002 through 2007, use integration to approximate the surplus revenue generated during those years.
- Will the models found in parts (a) and (b) intersect? Explain. Based on your answer and news reports about the fund, will these models be accurate for long-term analysis?

 84. **Lorenz Curve** Economists use *Lorenz curves* to illustrate the distribution of income in a country. A Lorenz curve, $y = f(x)$, represents the actual income distribution in the country. In this model, x represents percents of families in the country and y represents percents of total income. The model $y = x$ represents a country in which each family has the same income. The area between these two models, where $0 \leq x \leq 100$, indicates a country's "income inequality." The table lists percents of income y for selected percents of families x in a country.

x	10	20	30	40	50
y	3.35	6.07	9.17	13.39	19.45

x	60	70	80	90
y	28.03	39.77	55.28	75.12

- Use a graphing utility to find a quadratic model for the Lorenz curve.
- Plot the data and graph the model.
- Graph the model $y = x$. How does this model compare with the model in part (a)?
- Use the integration capabilities of a graphing utility to approximate the "income inequality."

- 85. Profit** The chief financial officer of a company reports that profits for the past fiscal year were \$893,000. The officer predicts that profits for the next 5 years will grow at a continuous annual rate somewhere between $3\frac{1}{2}\%$ and 5% . Estimate the cumulative difference in total profit over the 5 years based on the predicted range of growth rates.
- 86. Area** The shaded region in the figure consists of all points whose distances from the center of the square are less than their distances from the edges of the square. Find the area of the region.

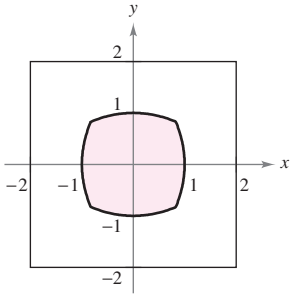


Figure for 86

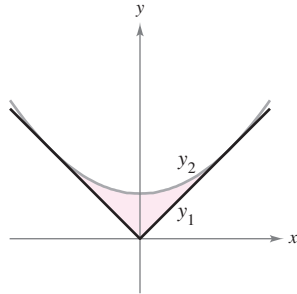
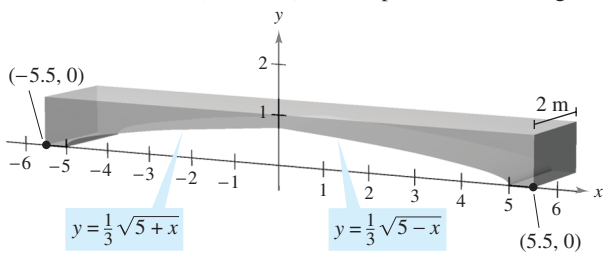


Figure for 87

- 87. Mechanical Design** The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$ (see figure).

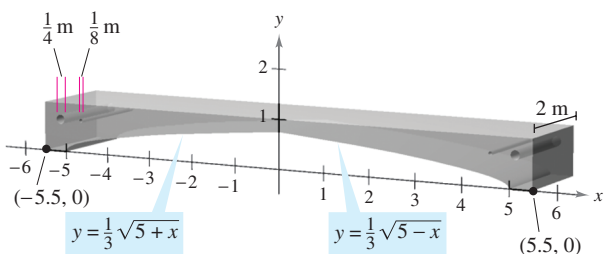
- (a) Find k if the parabola is tangent to the graph of y_1 .
- (b) Find the area of the surface of the machine part.

- 88. Building Design** Concrete sections for a new building have the dimensions (in meters) and shape shown in the figure.



- (a) Find the area of the face of the section superimposed on the rectangular coordinate system.
- (b) Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.
- (c) One cubic meter of concrete weighs 5000 pounds. Find the weight of the section.

- 89. Building Design** To decrease the weight and to aid in the hardening process, the concrete sections in Exercise 88 often are not solid. Rework Exercise 88 to allow for cylindrical openings such as those shown in the figure.



True or False? In Exercises 90–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- 90.** If the area of the region bounded by the graphs of f and g is 1, then the area of the region bounded by the graphs of $h(x) = f(x) + C$ and $k(x) = g(x) + C$ is also 1.

- 91.** If $\int_a^b [f(x) - g(x)] dx = A$, then $\int_a^b [g(x) - f(x)] dx = -A$.

- 92.** If the graphs of f and g intersect midway between $x = a$ and $x = b$, then

$$\int_a^b [f(x) - g(x)] dx = 0.$$

- 93. Area** Find the area between the graph of $y = \sin x$ and the line segments joining the points $(0, 0)$ and $(\frac{7\pi}{6}, -\frac{1}{2})$, as shown in the figure.

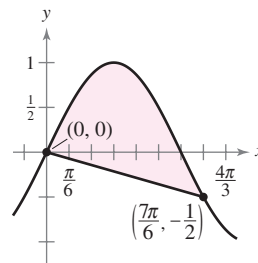


Figure for 93

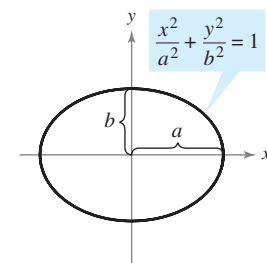


Figure for 94

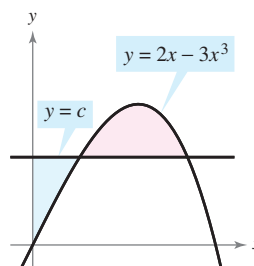
- 94. Area** Let $a > 0$ and $b > 0$. Show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab (see figure).

Putnam Exam Challenge

- 95.** The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as shown in the figure. Find c so that the areas of the two shaded regions are equal.



This problem was composed by the Committee on the Putnam Prize Competition.
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